Recurrence, p.a.p. and R-closed properties for flows and foliations

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Motivation

To understand codimension **TWO** foliations and dynamical systems.

Background

Remage(1962) [R]
$\exists$ R-**closed** homeomorphism on $\mathbb{S}^2$ has a minimal set which is not a circle but a **circloid**.

Herman(1986) [H]
$\exists$ R-**closed** C^∞ diffeomorphism of $\mathbb{S}^2$ has a **pseudo-circle** as a minimal set

Mason(1973) [M]
Each non-periodic orientation-preserving R-**closed** homeomorphism on $\mathbb{S}^2$ has exactly two fixed points and every non-degenerate orbit closure is a homology 1-sphere.

Let G be a flow or a homeomorphism on a compact Hausdorff space.
The following are equivalent:
1. G is pointwise almost periodic
2. each orbit closure of G is minimal.
3. The set of orbit closures of G is a decomposition.

Moreover, the dynamical system G satisfies the following relations:

R-closed
$\Downarrow$
pointwise almost periodic
$\Downarrow$
pointwise recurrent

Preliminaries

Topological dynamics
Let M be a topological space with a G-action, where $G = \mathbb{R}$ or $\mathbb{Z}$. Write $O(x) := \{t \cdot x \mid t \in G\}$.

Definition 1. *G is* **pointwise almost periodic** *if for any neighborhood U of each point x, there is a positive number K such that $O(x) \subseteq [0, K]U$, where $[0, K]U = \{g \cdot y \mid g \in [0, K], y \in U\}$.*

Definition 2. *G is* R-**closed** *if $R := \{(x, y) \mid y \in \overline{O_x}\}$ is closed w.r.t. the product topology.*

Definition 3. *G is* **pointwise recurrent** *if $x \in \alpha(x) \cap \omega(x)$ for each point $x \in X$, where $\alpha(x)$ (resp. $\omega(x)$) is an alpha (resp. omega) limit set of x.*

Definition 4. *We define the equivalence relation $\sim$ by $O \sim O' \Leftrightarrow \overline{O} = \overline{O'}$ for orbits O, O' of G. The quotient space is called the orbit class space and denoted by $M/\hat{G}$. Put $\hat{O} := \cup\{O' \mid \overline{O} = \overline{O'}\}$, called the* **orbit class** *of O.*

By a continuum we mean a compact connected metrizable space.

Definition 5. *A continuum $A \subset X$ is said to be* ***annular*** *if it has a neighborhood $U \subset X$ homeomorphic to an open annulus such that $U - A$ has exactly two components, both homeomorphic to annuli.*

Definition 6. *We say a subset $C \subset X$ is a* ***circloid*** *if it is an annular continuum and does not contain any strictly smaller annular continuum as a subset.*

Definition 7. *A minimal set $\mathcal{M}$ on a surface homeomorphism $f : S \to S$ is* **an extension of a Cantor set** *if there are a surface homeomorphism $F : S \to S$ and a surjective continuous map $p : S \to S$ which is homotopic to the identity such that $p \circ f = F \circ p$ and $p(\mathcal{M})$ is a Cantor set which is a minimal set of F.*

Main results

Surface homeomorphisms

Let f be an R-**closed** homeomorphism on an orientable connected closed surface M. Write $O(x) := \{f^n(x) \mid n \in \mathbb{Z}\}$ the orbit of x of f. Then the following statements hold [Y2]:

Case (genus$(M) > 1$)
Each minimal set is either a periodic orbit or an extension of a Cantor set.
In particular, it is not a circloid.

Case ($M = \mathbb{T}^2$)
One of the following statements holds:
1) f is minimal.
2) f is periodic (i.e. $f^k =$ id for some $k > 0$).
3) Each minimal set is a finite disjoint union of essential circloids.
4) There is a minimal set which is an extension of a Cantor set.

Case ($M = \mathbb{S}^2$)
Suppose that f is orientation-preserving (resp. reversing). One of the following statements holds:
1) f is periodic.
2) The minimal sets of f (resp. f^2) are exactly two fixed points and a family of (null-homotopic) circloids and the orbit class space $\mathbb{S}^2/\hat{f} \cong [0, 1]$.

Key ideas of the proofs
1. The union of circloids is open.
2. Each connected component of the boundaries of the set of circloids is an element of $\mathbb{S}^2/\hat{f}$.

On the other hand, the following dichotomy holds [Y]:

Proposition 1. *One of the following statements holds for a suspension flow v_f of f:*
1. Each orbit culousre of v_f is toral.
2.$\exists$minimal set which is not locally connected.

Recall that a subset U of a topological space is toral if U is some dimensional torus $\mathbb{T}^k$, and is locally connected if every point of U admits a neighbourhood basis consisting of open connected subsets.

Key ideas of the proof
1. f: R-closed $\Leftrightarrow$ f^k: R-closed for all $k \in \mathbb{Z}$.
2. f: pointwise almost periodic $\Leftrightarrow$ f^k: pointwise almost periodic for all $k \in \mathbb{Z}$.
3. Use characterizations of locally connected minimal sets for surface homeomorphisms [BNW]

Surface flows

Let v be a continuous vector field of a closed connected surface M. Then the following relations holds [Y2]:

pointwise recurrent
$\Updownarrow$
pointwise almost periodic
$\Updownarrow$
(MIN or CPT)

Key ideas of the proof
1. The union of closed orbits is open.
2. There are no exceptional minimal sets.

Foliations

Let $\mathcal{F}$ be a foliation on a compact manifold and $M/\mathcal{F}$ the leaf space. We define the equivalence relation $\sim$ by $L \sim L' \Leftrightarrow \overline{L} = \overline{L'}$ for leaves L, L'. The quotient space is called the (leaf) class space and denoted by $M/\hat{\mathcal{F}}$. Put $\hat{L} := \cup\{L' \in \mathcal{F} \mid \overline{L} = \overline{L'}\}$, called the **(leaf) class** of L. As dynamical systems, we can define R-**closed (resp. pointwise almost periodic, pointwise recurrent, non-wandering)** foliations in the similar fashion. Then the following statements hold [Y3]:

$\mathcal{F}$: R-closed $\Leftrightarrow$ $M/\hat{\mathcal{F}}$: Hausdorff (i.e. T_2).
$\mathcal{F}$: pointwise almost periodic $\Leftrightarrow$ $M/\hat{\mathcal{F}}$: T_1.
$\mathcal{F}$: compact $\Leftrightarrow$ $M/\mathcal{F}$: T_1.

Question
$L \in \mathcal{F}$ is proper $\Leftrightarrow$ $\hat{L}$ consists of a single leaf?
In particular,
$\mathcal{F}$: proper $\Leftrightarrow$ each class consists of a single leaf?
(i.e. **$\mathcal{F}$: proper $\Leftrightarrow$ $M/\mathcal{F}$: T_0?**)

Codimension one foliations

Let $\mathcal{F}$ be a codimension one foliation on a compact manifold. Then the following relations holds [Y3]:

pointwise almost periodic
$\Updownarrow$
R-**closed**
$\Updownarrow$
(MIN or CPT)

Codimension two foliations

We have the following inclusion relation [Y3]:

$\{$**MIN or CPT**$\} \subsetneq \{R$-**closed**$\}$

Example(A R-closed fol which is neither MIN nor CPT)
Considering S^2 as a unit sphere in R^3, let f be any irrational rotation on S^2 around the z-axis. Then the suspension foliation of f is an R-closed foliation which is neither MIN nor CPT

References

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