



# Recurrence, p.a.p. and R-closed properties for flows and foliations

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## Motivation

To understand codimension **TWO** foliations and dynamical systems.

## Background

### Remage(1962) [R]

$\exists$  **R-closed** homeomorphism on  $\mathbb{S}^2$  has a minimal set which is not a circle but a **circloid**.

### Herman(1986) [H]

$\exists$  **R-closed**  $C^\infty$  diffeomorphism of  $\mathbb{S}^2$  has a **pseudo-circle** as a minimal set

### Mason(1973) [M]

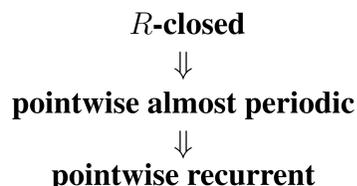
Each non-periodic orientation-preserving **R-closed** homeomorphism on  $\mathbb{S}^2$  has exactly two fixed points and every non-degenerate orbit closure is a homology 1-sphere.

Let  $G$  be a flow or a homeomorphism on a compact Hausdorff space.

The following are equivalent:

1.  $G$  is pointwise almost periodic
2. each orbit closure of  $G$  is minimal.
3. The set of orbit closures of  $G$  is a decomposition.

Moreover, the dynamical system  $G$  satisfies the following relations:



## Preliminaries

### Topological dynamics

Let  $M$  be a topological space with a  $G$ -action, where  $G = \mathbb{R}$  or  $\mathbb{Z}$ . Write  $O(x) := \{t \cdot x \mid t \in G\}$ .

**Definition 1.**  $G$  is **pointwise almost periodic** if for any neighborhood  $U$  of each point  $x$ , there is a positive number  $K$  such that  $O(x) \subseteq [0, K]U$ , where  $[0, K]U = \{g \cdot y \mid g \in [0, K], y \in U\}$ .

**Definition 2.**  $G$  is **R-closed** if  $R := \{(x, y) \mid y \in \overline{O_x}\}$  is closed w.r.t. the product topology.

**Definition 3.**  $G$  is **pointwise recurrent** if  $x \in \alpha(x) \cap \omega(x)$  for each point  $x \in X$ , where  $\alpha(x)$  (resp.  $\omega(x)$ ) is an alpha (resp. omega) limit set of  $x$ .

**Definition 4.** We define the equivalence relation  $\sim$  by  $O \sim O' \Leftrightarrow \overline{O} = \overline{O'}$  for orbits  $O, O'$  of  $G$ . The quotient space is called the orbit class space and denoted by  $M/\hat{\mathcal{F}}$ . Put  $\hat{O} := \cup\{O' \mid \overline{O} = \overline{O'}\}$ , called the **orbit class** of  $O$ .

By a continuum we mean a compact connected metrizable space.

**Definition 5.** A continuum  $A \subset X$  is said to be **annular** if it has a neighborhood  $U \subset X$  homeomorphic to an open annulus such that  $U - A$  has exactly two components, both homeomorphic to annuli.

**Definition 6.** We say a subset  $C \subset X$  is a **circloid** if it is an annular continuum and does not contain any strictly smaller annular continuum as a subset.

**Definition 7.** A minimal set  $\mathcal{M}$  on a surface homeomorphism  $f : S \rightarrow S$  is **an extension of a Cantor set** if there are a surface homeomorphism  $F : S \rightarrow S$  and a surjective continuous map  $p : S \rightarrow S$  which is homotopic to the identity such that  $p \circ f = F \circ p$  and  $p(\mathcal{M})$  is a Cantor set which is a minimal set of  $F$ .

## Main results

### Surface homeomorphisms

Let  $f$  be an **R-closed** homeomorphism on an orientable connected closed surface  $M$ . Write  $O(x) := \{f^n(x) \mid n \in \mathbb{Z}\}$  the orbit of  $x$  of  $f$ . Then the following statements hold [Y2]:

Case ( $\text{genus}(M) > 1$ )

**Each minimal set is either a periodic orbit or an extension of a Cantor set. In particular, it is not a circloid.**

Case ( $M = \mathbb{T}^2$ )

**One of the following statements holds:**

- 1)  $f$  is minimal.
- 2)  $f$  is periodic (i.e.  $f^k = \text{id}$  for some  $k > 0$ ).
- 3) Each minimal set is a finite disjoint union of **essential circloids**.
- 4) There is a minimal set which is an extension of a Cantor set.

Case ( $M = \mathbb{S}^2$ )

**Suppose that  $f$  is orientation-preserving (resp. reversing). One of the following statements holds:**

- 1)  $f$  is periodic.
- 2) The minimal sets of  $f$  (resp.  $f^2$ ) are exactly two fixed points and a family of (null-homotopic) **circloids** and the orbit class space  $\mathbb{S}^2/\hat{f} \cong [0, 1]$ .

### Key ideas of the proofs

1. The union of circloids is open.
2. Each connected component of the boundaries of the set of circloids is an element of  $\mathbb{S}^2/\hat{f}$ .

On the other hand, the following dichotomy holds [Y]:

**Proposition 1.** One of the following statements holds for a suspension flow  $v_f$  of  $f$ :

1. Each orbit closure of  $v_f$  is **toral**.
2.  $\exists$  minimal set which is **not locally connected**.

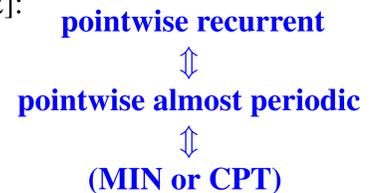
Recall that a subset  $U$  of a topological space is **toral** if  $U$  is some dimensional torus  $\mathbb{T}^k$ , and is **locally connected** if every point of  $U$  admits a neighbourhood basis consisting of open connected subsets.

### Key ideas of the proof

1.  $f$ : R-closed  $\Leftrightarrow f^k$ : R-closed for all  $k \in \mathbb{Z}$ .
2.  $f$ : pointwise almost periodic  $\Leftrightarrow f^k$ : pointwise almost periodic for all  $k \in \mathbb{Z}$ .
3. Use characterizations of locally connected minimal sets for surface homeomorphisms [BNW]

## Surface flows

Let  $v$  be a continuous vector field of a closed connected surface  $M$ . Then the following relations holds [Y2]:



### Key ideas of the proof

1. The union of closed orbits is open.
2. There are no exceptional minimal sets.

## Foliations

Let  $\mathcal{F}$  be a foliation on a compact manifold and  $M/\mathcal{F}$  the leaf space. We define the equivalence relation  $\sim$  by  $L \sim L' \Leftrightarrow \overline{L} = \overline{L'}$  for leaves  $L, L'$ . The quotient space is called the (leaf) class space and denoted by  $M/\hat{\mathcal{F}}$ . Put  $\hat{L} := \cup\{L' \in \mathcal{F} \mid \overline{L} = \overline{L'}\}$ , called the **(leaf) class** of  $L$ . As dynamical systems, we can define **R-closed (resp. pointwise almost periodic, pointwise recurrent, non-wandering)** foliations in the similar fashion. Then the following statements hold [Y3]:

$$\begin{array}{l} \mathcal{F} : \text{R-closed} \Leftrightarrow M/\hat{\mathcal{F}} : \text{Hausdorff (i.e. } T_2\text{)}. \\ \mathcal{F} : \text{pointwise almost periodic} \Leftrightarrow M/\hat{\mathcal{F}} : T_1. \\ \mathcal{F} : \text{compact} \Leftrightarrow M/\mathcal{F} : T_1. \end{array}$$

### Question

**$L \in \mathcal{F}$  is proper  $\Leftrightarrow \hat{L}$  consists of a single leaf?**  
In particular,  
 $\mathcal{F}$ : proper  $\Leftrightarrow$  each class consists of a single leaf?  
(i.e.  $\mathcal{F}$ : proper  $\Leftrightarrow M/\mathcal{F} : T_0$ ?)

## Codimension one foliations

Let  $\mathcal{F}$  be a codimension one foliation on a compact manifold. Then the following relations holds [Y3]:



## Codimension two foliations

We have the following inclusion relation [Y3]:

$$\{\text{MIN or CPT}\} \subsetneq \{\text{R-closed}\}$$

**Example(A R-closed fol which is neither MIN nor CPT)**  
Considering  $S^2$  as a unit sphere in  $R^3$ , let  $f$  be any irrational rotation on  $S^2$  around the  $z$ -axis. Then the suspension foliation of  $f$  is an R-closed foliation which is neither MIN nor CPT

## References

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