



Recurrence, p.a.p. and R-closed properties for flows and foliations

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Motivation

To understand codimension **TWO** foliations and dynamical systems.

Background

Remage(1962) [R]

\exists **R-closed** homeomorphism on \mathbb{S}^2 has a minimal set which is not a circle but a **circloid**.

Herman(1986) [H]

\exists **R-closed** C^∞ diffeomorphism of \mathbb{S}^2 has a **pseudo-circle** as a minimal set

Mason(1973) [M]

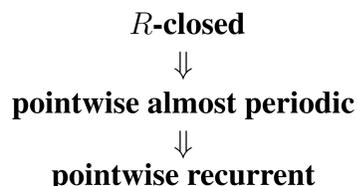
Each non-periodic orientation-preserving **R-closed** homeomorphism on \mathbb{S}^2 has exactly two fixed points and every non-degenerate orbit closure is a homology 1-sphere.

Let G be a flow or a homeomorphism on a compact Hausdorff space.

The following are equivalent:

1. G is pointwise almost periodic
2. each orbit closure of G is minimal.
3. The set of orbit closures of G is a decomposition.

Moreover, the dynamical system G satisfies the following relations:



Preliminaries

Topological dynamics

Let M be a topological space with a G -action, where $G = \mathbb{R}$ or \mathbb{Z} . Write $O(x) := \{t \cdot x \mid t \in G\}$.

Definition 1. G is **pointwise almost periodic** if for any neighborhood U of each point x , there is a positive number K such that $O(x) \subseteq [0, K]U$, where $[0, K]U = \{g \cdot y \mid g \in [0, K], y \in U\}$.

Definition 2. G is **R-closed** if $R := \{(x, y) \mid y \in \overline{O_x}\}$ is closed w.r.t. the product topology.

Definition 3. G is **pointwise recurrent** if $x \in \alpha(x) \cap \omega(x)$ for each point $x \in X$, where $\alpha(x)$ (resp. $\omega(x)$) is an alpha (resp. omega) limit set of x .

Definition 4. We define the equivalence relation \sim by $O \sim O' \Leftrightarrow \overline{O} = \overline{O'}$ for orbits O, O' of G . The quotient space is called the orbit class space and denoted by $M/\hat{\mathcal{F}}$. Put $\hat{O} := \cup\{O' \mid \overline{O} = \overline{O'}\}$, called the **orbit class** of O .

By a continuum we mean a compact connected metrizable space.

Definition 5. A continuum $A \subset X$ is said to be **annular** if it has a neighborhood $U \subset X$ homeomorphic to an open annulus such that $U - A$ has exactly two components, both homeomorphic to annuli.

Definition 6. We say a subset $C \subset X$ is a **circloid** if it is an annular continuum and does not contain any strictly smaller annular continuum as a subset.

Definition 7. A minimal set \mathcal{M} on a surface homeomorphism $f : S \rightarrow S$ is an **extension of a Cantor set** if there are a surface homeomorphism $F : S \rightarrow S$ and a surjective continuous map $p : S \rightarrow S$ which is homotopic to the identity such that $p \circ f = F \circ p$ and $p(\mathcal{M})$ is a Cantor set which is a minimal set of F .

Main results

Surface homeomorphisms

Let f be an **R-closed** homeomorphism on an orientable connected closed surface M . Write $O(x) := \{f^n(x) \mid n \in \mathbb{Z}\}$ the orbit of x of f . Then the following statements hold [Y2]:

Case ($\text{genus}(M) > 1$)

Each minimal set is either a periodic orbit or an extension of a Cantor set. In particular, it is not a circloid.

Case ($M = \mathbb{T}^2$)

One of the following statements holds:

- 1) f is minimal.
- 2) f is periodic (i.e. $f^k = \text{id}$ for some $k > 0$).
- 3) Each minimal set is a finite disjoint union of **essential circloids**.
- 4) There is a minimal set which is an extension of a Cantor set.

Case ($M = \mathbb{S}^2$)

Suppose that f is orientation-preserving (resp. reversing). One of the following statements holds:

- 1) f is periodic.
- 2) The minimal sets of f (resp. f^2) are exactly two fixed points and a family of (null-homotopic) **circloids** and the orbit class space $\mathbb{S}^2/\hat{f} \cong [0, 1]$.

Key ideas of the proofs

1. The union of circloids is open.
2. Each connected component of the boundaries of the set of circloids is an element of \mathbb{S}^2/\hat{f} .

On the other hand, the following dichotomy holds [Y]:

Proposition 1. One of the following statements holds for a suspension flow v_f of f :

1. Each orbit closure of v_f is **toral**.
2. \exists minimal set which is **not locally connected**.

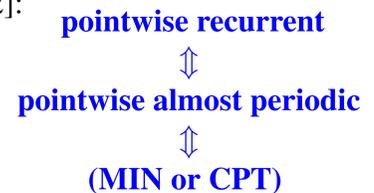
Recall that a subset U of a topological space is **toral** if U is some dimensional torus \mathbb{T}^k , and is **locally connected** if every point of U admits a neighbourhood basis consisting of open connected subsets.

Key ideas of the proof

1. f : R-closed $\Leftrightarrow f^k$: R-closed for all $k \in \mathbb{Z}$.
2. f : pointwise almost periodic $\Leftrightarrow f^k$: pointwise almost periodic for all $k \in \mathbb{Z}$.
3. Use characterizations of locally connected minimal sets for surface homeomorphisms [BNW]

Surface flows

Let v be a continuous vector field of a closed connected surface M . Then the following relations holds [Y2]:



Key ideas of the proof

1. The union of closed orbits is open.
2. There are no exceptional minimal sets.

Foliations

Let \mathcal{F} be a foliation on a compact manifold and M/\mathcal{F} the leaf space. We define the equivalence relation \sim by $L \sim L' \Leftrightarrow \overline{L} = \overline{L'}$ for leaves L, L' . The quotient space is called the (leaf) class space and denoted by $M/\hat{\mathcal{F}}$. Put $\hat{L} := \cup\{L' \in \mathcal{F} \mid \overline{L} = \overline{L'}\}$, called the **(leaf) class** of L . As dynamical systems, we can define **R-closed (resp. pointwise almost periodic, pointwise recurrent, non-wandering)** foliations in the similar fashion. Then the following statements hold [Y3]:

$$\begin{array}{l} \mathcal{F} : \text{R-closed} \Leftrightarrow M/\hat{\mathcal{F}} : \text{Hausdorff (i.e. } T_2\text{)}. \\ \mathcal{F} : \text{pointwise almost periodic} \Leftrightarrow M/\hat{\mathcal{F}} : T_1. \\ \mathcal{F} : \text{compact} \Leftrightarrow M/\mathcal{F} : T_1. \end{array}$$

Question

$L \in \mathcal{F}$ is proper $\Leftrightarrow \hat{L}$ consists of a single leaf?
In particular,
 \mathcal{F} : proper \Leftrightarrow each class consists of a single leaf?
(i.e. \mathcal{F} : proper $\Leftrightarrow M/\mathcal{F} : T_0$?)

Codimension one foliations

Let \mathcal{F} be a codimension one foliation on a compact manifold. Then the following relations holds [Y3]:



Codimension two foliations

We have the following inclusion relation [Y3]:

$$\{\text{MIN or CPT}\} \subsetneq \{\text{R-closed}\}$$

Example(A R-closed fol which is neither MIN nor CPT)
Considering S^2 as a unit sphere in R^3 , let f be any irrational rotation on S^2 around the z -axis. Then the suspension foliation of f is an R-closed foliation which is neither MIN nor CPT

References

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