METRIC DIFFUSION ALONG COMPACT FOLIATIONS

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Wasserstein distance

The Wasserstein distance $d_{\mathcal{W}}$ of Borel probability measures μ and ν on Polish space X (complete separable metric space) endowed with a metric d is defined by

$$d_{\mathcal{W}}(\mu,\nu) = \inf \int_{M\times M} d(x,y)d\rho$$

where infimum is taken over all Borel probability measures ρ on $X \times X$ satisfying

$$\rho(A\times X) = \mu(A), \quad \textit{and} \quad \rho(X\times B) = \nu(B).$$

for any measurable sets A, B of X.

Theorem

The set $\mathcal{P}(M)$ of all Borel probability measures with finite first moment endowed with $d_{\mathcal{W}}$ is a metric space. Moreover, $d_{\mathcal{W}}$ metrizes the weak-$*$ topology.

C. Villani, Topics in optimal transportation, American Mathematical Society, Providence, 2003.

Foliated heat diffusion

On compact foliated Riemannian manifold $(M,\mathcal{F},g)$ define the *foliated Laplace operator* $\Delta_{\mathcal{F}}$ by

$$\Delta_{\mathcal{F}} f(x) = \Delta_{L_x} f(x), \quad x\in M,$$

where L_x is a leaf through x, and Δ_L is Laplace operator on $(L, g|L)$. The operator $\Delta_{\mathcal{F}}$ acts on bounded measurable functions, which are C^2-smooth along the leaves.

Associate with the operator $\Delta_{\mathcal{F}}$ the one-parameter semi-group D_t, $t\geq 0$, of *heat diffusion operators* characterized by

$$D_0 = \mathrm{id}, \ D_{t+s} = D_t \circ D_s, \ \frac{d}{dt}D_t|_{t=0} = \Delta_{\mathcal{F}}.$$

D_t restricted to a leaf $L\in\mathcal{F}$ coincides with the heat diffusion operators on L, which are given by

$$D_t f(x) = \int_{L_x} f(y)p(x,y;t)d\mathrm{vol}_{L_x},$$

where $p(\cdot,\cdot;t)$ is a *foliated heat kernel* on $(M,\mathcal{F})$.

A probability measure μ on $(M,\mathcal{F},g)$ is *harmonic* if for any $f: M\to\mathbb{R}$

$$\int_M \Delta_{\mathcal{F}} f d\mu = 0.$$

Theorem

On any compact foliated Riemannian manifold, harmonic probability measure exist.

L. Garnett, Foliations, the Ergodic Theorem and Brownian Motions, J. Func. Anal. 51 (1983), no. 3, 285-311.

Let μ be a probability measure on M. Define *the diffused measure* $D_t\mu$ by the formula

$$\int_M f D_t\mu = \int_M D_t f d\mu,$$

where f is any bounded measurable function on M. A measure μ is called *diffusion invariant* when $D_t\mu = \mu$.

Metric diffusion

Definition Let $(M,\mathcal{F},g)$ be a smooth compact foliated manifold equipped with a Riemannian metric g carrying foliation $\mathcal{F}$. The *metric diffused along* $\mathcal{F}$ is defined by

$$\mathrm{D}_t d(x,y) = d_{\mathcal{W}}(\mathrm{D}_t\delta_x, \mathrm{D}_t\delta_y)$$

where d_z denotes the Dirac measure concentrated in the point z.

Theorem

For any $s,t\geq 0$, metrics $\mathrm{D}_t d$ and $D_s d$ are equivalent.

Sz. Walczak, Metric diffusion along compact foliations, in preparation.

Metric diffusion along compact foliations of dimension 1

The natural isometric embedding ι_t of $(M, D_t d)$ into the space of all probability measures on M (denoted by $\mathcal{P}(M)$) equipped with the Wasserstein distance is defined by

$$\iota_t : M \ni x \mapsto D_t\delta_x \in \mathcal{P}(M).$$

This allows us to study the Hausdorff distance of the embeddings $\iota_t(M)$ and $\iota_s(M)$, and to study the limit $\lim_{t\to\infty}\iota_t(M)$.

Let $L, L'\in\mathcal{F}$ be two leaves. One can define a metric ρ_{vol} in the space of leaves $\mathcal{L}$ by $\rho_{\mathrm{vol}}(L,L') = d_{\mathcal{W}}(\overline{\mathrm{vol}}(L), \overline{\mathrm{vol}}(L'))$, where $\overline{\mathrm{vol}}(F)$ denotes the normalized volume of a leaf.

Theorem

Let $(M,\mathcal{F},g)$ be a compact foliated Riemannian manifold carrying a compact foliation $\mathcal{F}$ with empty bad set. The Hausdorff limit $\lim_{t\to\infty}\iota_t(M)$ of a diffused metric is isometric to the space of leaves $\mathcal{L}$ equipped with the metric ρ_{vol}.

Sz. Walczak, Metric diffusion along compact foliations, in preparation.

Example For a topological group G and a close curve $\gamma : [0,2\pi] \to G$ define a one dimensional foliation $\mathcal{F}(\gamma)$ on $S^1\times G$ by filling it with closed curves

$$[0,2\pi] \ni s \mapsto (s, \gamma(s)\gamma(t)^{-1}x) \in S^1\times G.$$

Let $\tau\in(0,1]$, and let $\gamma_\tau : [0,2\pi]\to S^3$ be as follows:

∘ if $\tau = \frac{1}{2n+1} - t$, $0\leq t\leq \frac{1}{(2n+1)(2n+2)} = a_n$, $n = 0,1,2,\ldots$ then

$$\gamma_\tau(s) = (\sqrt{1-(\frac{t}{a_n})^2}\, e^{ins}, \frac{t}{a_n}\, e^{ins}), \quad s\in[0,2\pi];$$

∘ if $\tau = \frac{1}{2n} - t$, $0\leq t\leq \frac{1}{2n(2n+1)} = b_n$, $n=1,2,\ldots$ then

$$\gamma_\tau(s) = (\frac{t}{b_n}\, e^{ins}, \sqrt{1-(\frac{t}{b_n})^2}\, e^{i(n+1)s}), \quad s\in[0,2\pi].$$

For given $\tau\in(0,1]$, foliate the set $\{\tau\}\times S^1\times S^3$ by $\mathcal{F}(\gamma_\tau)$. Complement the foliation $\tilde{\mathcal{F}}$ of $M = [0,1]\times S^1\times S^3$ by

$$\{0\}\times\{t\}\times\Phi\cdot g, \quad g\in S^3, t\in S^1$$

with $\Phi = \{(e^{is},0), s\in[0,2\pi]\}$ on $\{0\}\times S^1\times S^3$. $\tilde{\mathcal{F}}$ has non-empty bad set.

Let $h : [0,2\pi]\to[0,2\pi]$ be a increasing function with two local maxima. For a smooth homotopy $\bar{h} : [0,1]\times[0,2\pi]\to[0,2\pi]$ from identity to h, define $\tilde{h} : [0,1]\times[0,2\pi]\to[0,2\pi]$ by

$$\tilde{h}(t,s) = \begin{cases} \bar{h}(2t,s), & t\in[0,\frac{1}{2}], \\ \bar{h}(-2t+2,s), & t\in[\frac{1}{2},1]. \end{cases}$$

Define $H_n : [0,1]\times S^1\times S^3 \to [0,1]\times S^1\times S^3$ by

$$H_n(\tau,s,x) = \begin{cases} (\tau, \tilde{h}(n(n+1)\tau - n), s), x), & (\tau,s,x)\in[\frac{1}{2n+2},\frac{1}{2n+1}]\times S^1\times S^3, \\ (\tau,s,x) & \textit{otherwise}. \end{cases}$$

Modify $\tilde{\mathcal{F}}$ as follows: For $n_1 = 1$ set $\mathcal{F}_1 = H_1(\tilde{\mathcal{F}})$. Next, choose $\theta_1 > 0$ such that for all $\theta > \theta_1$ and all $p = (\tau,s,x)\in[\frac{1}{2n_1+2},1]\times S^1\times S^3$

$$d_{\mathcal{W}}(D_{\theta_1}\delta_p, \overline{\mathrm{vol}}(L_p)) < \frac{1}{2^{n_1}}.$$

Suppose that we have chosen $n_k > n_{k-1}$ and $\theta_k > \theta_{k-1}$ such that for foliation

$$\mathcal{F}_k = (H_k\circ\cdots\circ H_1)(\tilde{\mathcal{F}})$$

and all $p = (\tau,s,x)\in[\frac{1}{2(n_k+1)},1]\times S^1\times S^3$

$$d_{\mathcal{W}}(D_{\theta_k}\delta_p, \overline{\mathrm{vol}}(L_p)) < \frac{1}{2^{n_k}}.$$

Let us choose $n_{k+1} > n_k$ for which all leaves of $\mathcal{F}_{k+1} = H_{k+1}(\mathcal{F}_k)$ passing through $p = (\tau,s,x)\in[0,\frac{1}{n_{k+1}}]\times S^1\times S^3$ satisfy

$$d_{\mathcal{W}}(D_{\theta_k}\delta_p, \overline{\mathrm{vol}}(L_{(0,s,x)})) < \frac{1}{2^k}.$$

Finally, define foliation $\mathcal{F}$ on $[0,1]\times S^1\times S^3$ as $(\cdots H_n\circ\cdots\circ H_1)(\tilde{\mathcal{F}})$ and equip $M = [0,1]\times S^1\times S^3$ with the Riemannian metric d induced from $\mathbb{R}^7$.

Theorem

The family $\iota_t(M)$ does not satisfies the Cauchy condition, therefore it does not converge in the Hausdorff metric.

Sz. Walczak, Metric diffusion along compact foliations, in preparation.

Final remarks

The most crucial question that can be asked is the one about the convergence of an arbitrary compact diffused foliation of dimension one. Since there are some lacks in the proofs, we only formulate the following.

Hypothesis

Let $(M,\mathcal{F},g)$ be a compact foliated Riemannian manifold carrying a 1-dimensional compact foliation $\mathcal{F}$. If the limit $\lim_{t\to\infty}\iota_t(M)$ exists, then the bad set of $\mathcal{F}$ is empty.

Bibliography

1. L. Garnett, *Foliations, the Ergodic Theorem and Brownian Motions*, J. Func. Anal. **51** (1983), no. 3, 285-311.
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