Dehn surgeries along $(-2, 3, 2s+1)$-type Pretzel knot with no $\mathbb{R}$-covered foliation and left-orderable groups

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1. Main theorem and background

We will present a result which is an extension of following theorem of J.Jun[Jun].

Theorem (J.Jun [Jun], 2004)

Let K be the $(-2, 3, 7)$-Pretzel knot in S^3 and $E_K(p/q)$ denotes the closed 3-manifold obtained by (p, q)-Dehn surgery along K. If $p/q \geqq 10$ and p is odd, then $E_K(p/q)$ does not contain an $\mathbb{R}$-covered foliation.

Our result is following.

Main Theorem [N]

Let K be a $(-2, 3, 2s+1)$-type Pretzel knot ($s \geqq 3$). If $p/q \geqq 4s+7$ and p is odd, then $E_K(p/q)$ does not contain an $\mathbb{R}$-covered foliation.

Note that in the case of $s = 3$, our result is contained in the Jun's result, but it has been somewhat worse with respect to an estimation of slopes.
These theorems are proved by following ideas:

Reebless foliations and the action of π_1 on its leaf space

- Let $\mathcal{F}$ be a Reebless foliation in a closed 3-manifold M, and $\tilde{\mathcal{F}}$ be a lift in the universal cover $\tilde{M}$. The quotient space $T = \tilde{M}/\tilde{\mathcal{F}}$ is called the leaf space.
- An open transversal of leaves gives an 1-manifold structure, and then T is a simply connected 1-manifold, but it might be non-Hausdorff.
- There is a natural action of $\pi_1(M)$ on T induced by the action on $\tilde{M}$.
- Since $\mathcal{F}$ has no Reeb component, this action has no global fixed point.

Therefore, if there is no nontrivial π_1-action on any leaf space T, then M does not contain Reebless foliations. The case when the leaf space T is non-Hausdorff has very complicated situations to consider the actions of π_1. We now focus on the case that $T \cong \mathbb{R}$. In this case $\mathcal{F}$ is called an $\mathbb{R}$-covered foliation.
In order to investigate these action, we have to obtain an explicit presentation of π_1 and transform it into a appropriate form. It will be mentioned in the next section.

2. fundamental group - a good presentation

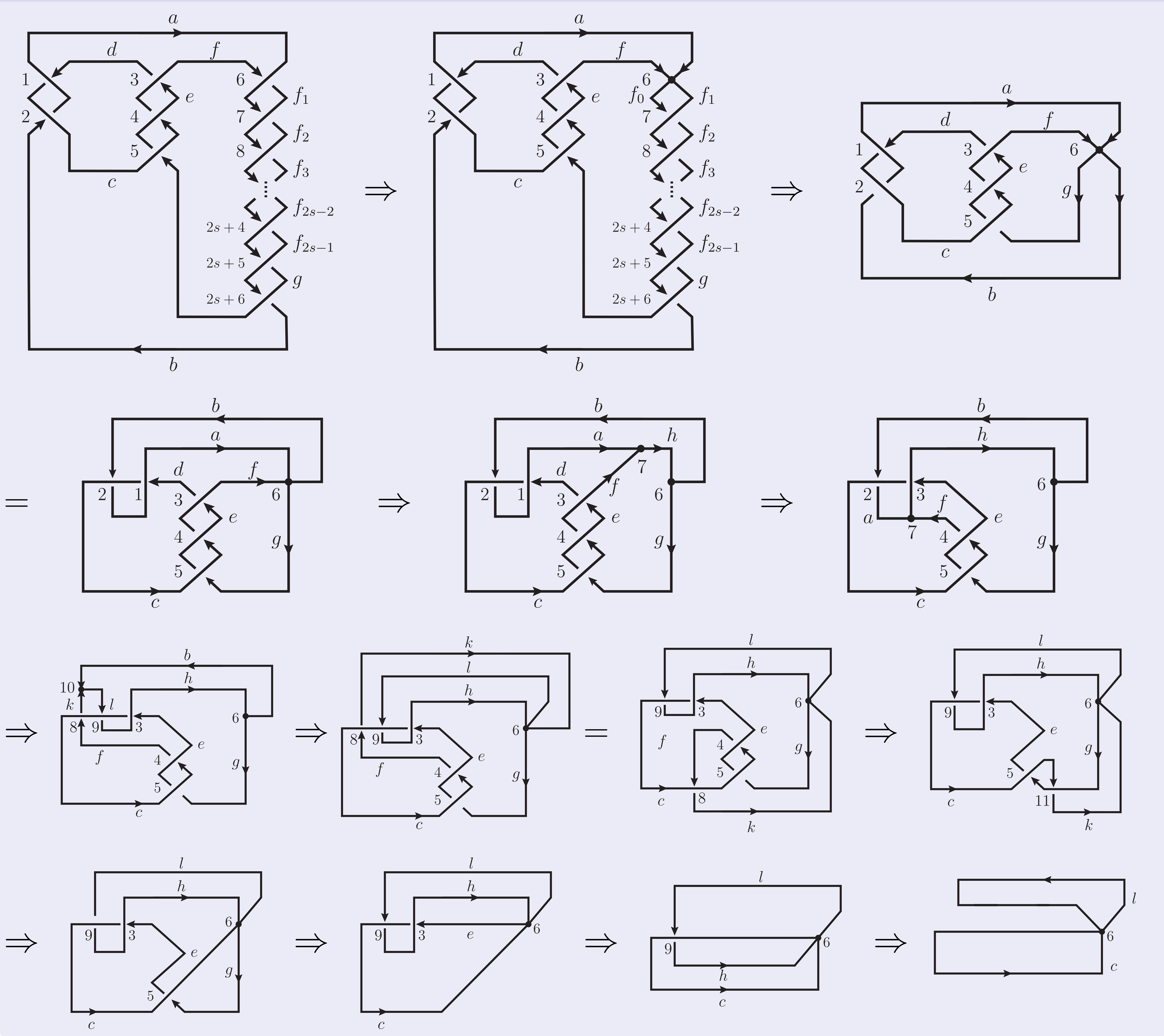
On a presentation of fundamental groups

- In [Jun], the presentations of the fundamental group of $(-2, 3, 7)$-Pretzel knot and its meridian-longitude pair are obtained by using SnapPea.
- In the cases of $(-2, 3, 2s+1)$, we want to obtain these presentations without using SnapPea.
- Moreover, since we want to investigate actions on a leaf space T, its presentation is better to be simple, especially we require it has two generators and one relator.

How to obtain a good presentation of the fundamental group

- $(-2, 3, 2s+1)$-Pretzel knot K_s is a tunnel number one knot by [MSY], then there should be a presentation of $\pi_1(K)$ with two generators and one relator.
- In [HTT], they presented a way for obtaining such presentation with a tunnel number one knot.
- They start from the Wirtinger presentation of a knot. The presentation is transformed by a sequence of Tietze transformations, and its process is along the path of deformations starting from the initial graph which obtained by collapsing one crossing of the knot, and it will approach to the shape $S^1 \vee S^1$.
- We use local moves to deform a graph in above process. Each moves corresponds to one of Tietze transformations.

$(-2, 3, 2s+1)$-Pretzel knot and the sequence of desired deformations



How to obtain a presentation of the meridian-longitude pair

- In order to obtain a presentation of $G_{K_s}(p, q) = \pi_1(E_{K_s}(p/q))$, we have to get a presentation of a meridian-longitude pair.
- We first fix a meridian c and get a presentation of a longitude L_1 which are compatible with the Wirtinger presentation by using the method which appeared in the book of Burde and Zieschang.
- Then we continue to modify L_i from $i = 1$ along the steps of sequence of Tietze transformations.
- By modifying the last presentation of the longitude slightly, we finally obtain
$$L = \bar{c}^{2s-2} l c l^s c l^s c l \bar{c}^{2s+9}.$$

The obtained presentation of $\pi_1(E_{K_s})$ and its meridian-longitude pair

$$G_{K_s} = \langle c, l \mid c l \bar{c} \bar{l}^s \bar{c} \bar{l} c l c l^{s-1} \rangle,$$
$$\text{meridian } M = c, \ \text{ longitude } L = \bar{c}^{2s-2} l c l^s c l^s c l \bar{c}^{2s+9}$$
$$G_{K_s}(p, q) := \pi_1(E_K(p/q)) = \langle c, l \mid c l \bar{c} \bar{l}^s \bar{c} \bar{l} c l c l^{s-1}, M^p L^q \rangle$$

3. Proof of main theorem

We prove main theorem as an analogy of the proof in [Jun] and [RSS]. Please see my paper [N] for details.

4. Problems and related topics

Problems

- Does this result extend to the case of $(-2, 2r+1, 2s+1)$-Pretzel knot?
  $\Rightarrow$ We are going to calculate necessary presentations by using the presentations appeared in [Tran] in cooperate with him.
- Does it extend to the case of a Reebless foliation?
  $\Rightarrow$ We have already get the good presentation of the fundamental group, then we can try prove the case of a Reebless foliation by using the same method of [RSS] and [Jun]. But it is very complicated.

Related Topics in the viewpoint of Dehn surgery

- Does this result be useful for problems of an exceptional surgery?
  $\Rightarrow$ The exceptional surgery along these type Pretzel knot is determined by [IJ], and we can see that this result for $\mathbb{R}$-covered foliations is of not use for these exceptional surgery as following.

  Corollary

  There are infinitely many pretzel knots which does not admit finite or cyclic surgery, but they admit Dehn surgery which produces a closed manifold which cannot contain an $\mathbb{R}$-covered foliation.
- However, we think that there is possibility that it is of use for a problem of cosmetic surgery [Ichihara].

Related Topics in the viewpoint of Left-orderable group

- A group G is left-orderable if there exists a total ordering $<$ of the elements of G such that for any elements f, g, h of G, if $f < g$ then $hf < hg$.
- It is known that a countable group G is left-orderable if and only if there exists a faithful action of G on $\mathbb{R}$, that is, there is no point of $\mathbb{R}$ which fixed by any element of G.
- By this fact, if a closed 3-manifold M contains an $\mathbb{R}$-covered foliation, the fundamental group of M is left-orderable. Then we can conclude the following corollary.

  Corollary

  $G = G_{K_s}(p, q)$ denotes the fundamental group of the closed manifold which obtained by Dehn surgery along K_s with slope p/q. If $q > 0$, $p/q \geqq 4s+7$ and p is odd, G is not left-orderable.
- Clay and Watson showed the following theorem in [CW] related to L-space conjecture [BGW].

  Theorem (Clay and Watson [CW], 2012)

  Let K_m be a $(-2, 3, 2m+5)$-type Pretzel knot. If $p/q > 2m+15$ and $m \geqq 0$, the fundamental group $\pi_1(E_{K_m}(p/q))$ is not left-orderable.
- There are many interaction between a study of $\mathbb{R}$-covered foliations and a study of left-orderability of the fundamental group of a closed 3-manifold, so we think that these objects will be more interesting.

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