On the homeomorphism and diffeomorphism groups fixing a point

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Introduction

Let M be a topological metrizable manifold and let $\mathcal{H}(M)$ be the identity component of the group of all compactly supported homeomorphisms of M. By $\mathcal{H}(M,p)$, where $p \in M$, we denote the identity component of the group of all $h \in \mathcal{H}(M)$ with $h(p) = p$.

- A group G is called **perfect** if it is equal to its own commutator subgroup $[G,G]$, that is $H_1(G) = 0$.
- A manifold M admits a **compact exhaustion** iff there is a sequence $\{M_i\}_{i=1}^{\infty}$ of compact submanifolds with boundary such that $M_1 \subset \operatorname{Int} M_2 \subset M_2 \subset \dots$ and $M = \bigcup_{i=1}^{\infty} M_i$.

Theorem 1 [3]

Assume that either M is compact (possibly with boundary), or M is noncompact boundaryless and admits a compact exhaustion. Then $\mathcal{H}(M)$ is perfect.

The proof of Theorem 1 is a consequence of J.N.Mather's paper combined with results of R.D.Edwards and R.C.Kirby. A special case of Theorem 1 was already showed by G.M.Fisher.

- A group is called **bounded** if it is bounded with respect to any bi-invariant metric.
- For $g \in [G,G]$ the least k such that g is a product of k commutators is called the **commutator length** of g and is denoted by $\mathrm{cl}_G(g)$. For any perfect group G denote by cld_G the commutator length diameter of G, i.e. $\mathrm{cld}_G := \sup_{g \in G} \mathrm{cl}_G(g)$.
- A group G is called **uniformly perfect** if G is perfect and $\mathrm{cld}_G < \infty$.
- Let G be a group. A **conjugation-invariant norm** on G is a function $\nu : G \to [0,\infty)$ for every $g,h \in G$ we have
  a) $\nu(g) > 0$ if and only if $g \neq e$,
  b) $\nu(g^{-1}) = \nu(g)$,
  c) $\nu(gh) \leqslant \nu(g) + \nu(h)$,
  d) $\nu(hgh^{-1}) = \nu(g)$.

It is easy to see that G is bounded if and only if any conjugation-invariant norm on G is bounded.
Observe that the commutator length cl_G is a conjugation-invariant norm on $[G,G]$, or on G if G is a perfect group.

Proposition 2

Let G be perfect and bounded group. Then G is uniformly perfect.

Main results

Theorem 3

- The groups $\mathcal{H}(\mathbb{R}^n, 0)$ and $\mathcal{H}(\mathbb{R}^n_+, 0)$ are perfect, where $\mathbb{R}^n_+ = [0,\infty) \times \mathbb{R}^{n-1}$.
- Assume that either M is compact (possibly with boundary), or M is noncompact boundaryless and admits a compact exhaustion. Then the group $\mathcal{H}(M,p)$ is perfect.

A similar result was obtained by T.Tsuboi in [5]. He proved that $\mathcal{H}([0,1])$ is perfect by using different argument than that for Theorem 3. Next he generalized the result for Lipschitz homeomorphisms and for C^1-diffeomorphisms (resp. C^∞-diffeomorphisms) tangent (resp. infinitely tangent) to the identity at the endpoints. Observe as well that Theorem 3 was proved for M closed by K.Fukui in [2]. However, our proof is different than that in [2] and it leads to following theorem.

Theorem 4

The group $\mathcal{H}(\mathbb{R}^n, 0)$ is uniformly perfect and its commutator length diameter is less or equal 2. The same is true for $\mathcal{H}(\mathbb{R}^n_+, 0)$.

Let $\mathcal{D}^r(M)$ (resp. $\mathcal{D}^r(M,p)$) be the identity component of the group of all compactly supported C^r-diffeomorphisms of M (resp. fixing $p \in M$). It is easy to see that $\mathcal{D}^r(M,p)$ is not perfect for $r \geqslant 1$. Moreover, K.Fukui calculated that $H_1(\mathcal{D}^\infty(\mathbb{R}^n, 0)) = \mathbb{R}$.

Theorem 5

- $\mathcal{H}(\mathbb{R}^n, 0)$ is bounded group.
- Assume that either M is compact (possibly with boundary), or M is noncompact boundaryless and admits a compact exhaustion. Then the group $\mathcal{H}(M)$ is bounded whenever $\mathcal{H}(M,p)$ is bounded.

Note that this theorem is no longer true in the C^r category for $r \geqslant 1$. Choose a chart at p. Then there is the epimorphism $\mathcal{D}^r(M,p) \ni f \mapsto \mathrm{jac}_p f \in \mathbb{R}_+$, where $\mathrm{jac}_p f$ is the jacobian of f at p in this chart. From Proposition 1.3 in [1] an abelian group is bounded if and only if it is finite and Lemma 1.10 in [1] implies that $\mathcal{D}^r(M,p)$ is unbounded.

References

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