Some remarks on the reconstruction problem of symplectic and cosymplectic manifolds

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What is the reconstruction problem?

Let M and N be manifolds and let $G \leqslant \mathrm{Aut}(M)$ and $H \leqslant \mathrm{Aut}(N)$ be some automorphisms groups on M and N respectively. For which M, N, G, H if there is an isomorphism $\varphi : G \longrightarrow H$ then there is also a homeomorphism $\tau : M \longrightarrow N$ such that $\varphi(g) = \tau g \tau^{-1}$ for any $g \in G$? Moreover, when τ is a diffeomorphism preserving the structure?

M. Rubin's results

Theorem 1 [Rubin]
Let X and Y be regular topological spaces and let $H(X)$, $H(Y)$ denote groups of all homeomorphisms on X and Y respectively. Let $G \leqslant H(X)$ and $H \leqslant H(Y)$ be factorizable and non-fixing. Assume that there is an isomorphism $\varphi : G \to H$. Then there is a unique homeomorphism $\tau : X \to Y$ such that $\varphi(g) = \tau g \tau^{-1}$ for any $g \in G$.

Theorem 2 [Rubin]
Let X and Y be regular topological spaces and let $G \leqslant H(X)$, $H \leqslant H(X)$. Assume that

- there are $G_1 \leqslant G$ and $H_1 \leqslant H$ such that G_1, H_1 are factorizable and non-fixing groups on X and Y respectively,
- for every $x \in X$ $\mathrm{int}(\overline{G(x)}) \neq \emptyset$ and for every $y \in Y$ $\mathrm{int}(\overline{H(y)}) \neq \emptyset$.

Suppose that there is a group isomorphism $\varphi : G \longrightarrow H$. Then there is a homeomorphism $\tau : X \longrightarrow Y$ such that $\varphi(g) = \tau g \tau^{-1}$ for any $g \in G$.

Some definitions and notation

- Factorization property
  A group $G \leqslant H(X)$ is called factorizable if for any $g \in G$ and for any open covering $\mathcal{U}$ of X there are $U_1, \dots, U_k \in \mathcal{U}$ and $g_1, \dots, g_k \in G$ such that $g = g_1 \dots g_k$ and $\mathrm{supp}(g_i) \subset U_i$ for each i .
- Non-fixing property
  A group $G \leqslant H(X)$ is called non-fixing if for any $x \in X$ there is $g \in G$ such that $g(x) \neq x$.
- Let (M, ω) be a symplectic manifold. Then the symbol $\mathrm{Symp}(M, \omega)$ will stand for the group of all symplectomorphisms on (M, ω).
- Let (M, θ, ω) be a cosymplectic manifold. Then symbols $\mathrm{Cosymp}(M, \theta, \omega)$, $\mathrm{Ham}(M, \theta, \omega)$, $\mathrm{Grad}(M, \theta, \omega)$ and $\mathrm{Ev}(M, \theta, \omega)$ will stand for the groups of all cosymplectomorphisms or hamiltonian, gradient and evolution cosymplectomorphisms respectively.
- In both symplectic and cosymplectic cases if G is a group then G_c denotes its subgroup of all compactly supported elements and G_0 denotes its subgroup of all elements that are isotopic with the identity.

Cosymplectic case

Lemma

Groups $\mathrm{Ham}_c(M, \theta, \omega)$ and $\mathrm{Grad}_c(M, \theta, \omega)$ are factorizable and non-fixing.

Corollary

Let $(M_1, \theta_1, \omega_1)$ and $(M_2, \theta_2, \omega_2)$ be cosymplectic manifolds and let $G(M_i) = \mathrm{Ham}_c(M_i, \theta_i, \omega_i)$ or $G(M_i) = \mathrm{Grad}_c(M_i, \theta_i, \omega_i)$. If there is an isomorphism $\varphi : G(M_1) \longrightarrow G(M_2)$ then there is a unique homeomorphism $\tau : M_1 \longrightarrow M_2$ such that for any $g \in G(M_1)$ one have $\varphi(g) = \tau g \tau^{-1}$.

Main Theorem

Let $(M_1, \theta_1, \omega_1)$ and $(M_2, \theta_2, \omega_2)$ be cosymplectic manifolds. Let $G(M_i) = \mathrm{Cosymp}(M_i, \theta_i, \omega_i)$ or $G(M_i) = \mathrm{Ev}(M_i, \theta_i, \omega_i)$ or $G(M_i) = \mathrm{Grad}(M_i, \theta_i, \omega_i)$. If there exists an isomorphism $\varphi : G(M_1) \longrightarrow G(M_2)$ then there is a unique diffeomorphism $\tau : M_1 \longrightarrow M_2$ such that for any $g \in G(M_1)$ there is $\varphi(g) = \tau g \tau^{-1}$ and $\tau_* \omega_1 = \lambda \omega_2$ for some nowhere vanishing $\lambda \in \mathcal{C}^\infty(M_2)$ constant on leaves of symplectic foliation.

Theorem

Let $(M_1, \theta_1, \omega_1)$ and $(M_2, \theta_2, \omega_2)$ be cosymplectic manifolds with complete Reeb vector fields. Let

$$\tau : (M_1, \theta_1, \omega_1) \longrightarrow (M_2, \theta_2, \omega_2)$$

be a homeomorphism such that
$\tau h \tau^{-1} \in \mathrm{Cosymp}(M_2) \Leftrightarrow h \in \mathrm{Cosymp}(M_1)$ or
$\tau h \tau^{-1} \in \mathrm{Grad}(M_2, \theta_2, \omega_2) \Leftrightarrow h \in \mathrm{Grad}(M_1, \theta_1, \omega_1)$ or
$\tau h \tau^{-1} \in \mathrm{Ev}(M_2, \theta_2, \omega_2) \Leftrightarrow h \in \mathrm{Ev}(M_1, \theta_1, \omega_1)$.
Then τ is a $\mathcal{C}^\infty$ diffeomorphism.

The above theorem is an extension of Theorem of Takes to the cosymplectic case.

Symplectic case

In symplectic case it is known following theorem of A.Banyaga:

Theorem [Banyaga]
Let (M_i, ω_i) for $i = 1, 2$ be symplectic manifolds of dimension 2n. Assume that M_i are both compact or symplectic pairing for ω_i is identically 0. Let $G(M_i) = \mathrm{Symp}(M_i, \omega_i)$ or $G(M_i) = \mathrm{Symp}(M_i, \omega_i)_0$.
If $\varphi : G(M_1) \longrightarrow G(M_2)$ is an isomorphism then there is a unique diffeomorphism $\tau : M_1 \longrightarrow M_2$ such that $\varphi(g) = \tau g \tau^{-1}$ for any $g \in G(M_1)$ and $\tau_* \omega_1 = \lambda \omega_2$ for some constant $\lambda \in \mathcal{C}^\infty(M_2)$.

Our result is an extension of above theorem by omitting an assumption of compactness of M_i as well as an assumption of vanishing symplectic pairing for ω_i.

Main Theorem

Let (M_i, ω_i) for $i = 1, 2$ be symplectic manifolds and let $\varphi : \mathrm{Symp}(M_1, \omega_1) \longrightarrow \mathrm{Symp}(M_2, \omega_2)$ or $\varphi : \mathrm{Symp}(M_1, \omega_1)_0 \longrightarrow \mathrm{Symp}(M_2, \omega_2)_0$ be an isomorphism. Then there is a unique diffeomorphism $\tau : M_1 \longrightarrow M_2$ such that $\varphi(g) = \tau g \tau^{-1}$ for any $g \in \mathrm{Symp}(M_1, \omega_1)$ and $\tau_* \omega_1 = \lambda \omega_2$ for some constant $\lambda \in \mathcal{C}^\infty(M_2)$.

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