Configuration spaces of linkages on Riemannian surfaces

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Introduction

A *mechanical linkage* is a mechanism made of rigid rods linked together by flexible joints. Mathematically, we consider a linkage as graph whose edges have a fixed length. Some vertices are pinned down while others may move. A *realization* of a linkage $\mathcal{L}$ on a semi-Riemannian manifold $\mathcal{M}$ is a function which sends each vertex of the graph to a point of $\mathcal{M}$, respecting the lengths of the edges. The *configuration space* of a linkage $\mathcal{L}$, written $\mathcal{E}(V, \mathcal{M})$, is the set of all its realizations. Intuitively, the configuration space is the set of all the possible states of the mechanical linkage.
Many papers deal with linkages on the Euclidean plane $\mathbb{R}^2$, but the Definition of linkages extends naturally to any semi-Riemannian manifold.
On the Euclidean plane, Kempe [Kem75] has shown in 1875 that for any algebraic curve $\mathcal{C}$, for any euclidian ball $\mathcal{B} \subseteq \mathbb{R}^2$, there exists a linkage $\mathcal{L}$, and one vertex of this linkage v such that $\mathcal{C} \cap \mathcal{B}$ is exactly the set of the possible positions of v (his proof was flawed, but there is a rather simple way to make it correct, see [Abb08]). In particular, the famous *Peaucellier-Lipkin straight-line motion linkage* (Figure 1) forces a vertex to move on a straight line.
More recently, Kapovich and Millson [KM02] have shown that for any smooth compact manifold without boundary M, there exists a linkage for which the configuration space is diffeomorphic to a finite disjoint union of copies of M. Jordan and Steiner proved a weaker version of this theorem with more elementary techniques [JS99]. Thurston already gave lectures on a similar theorem in the 1980's but never wrote a proof.
When we consider the same linkage on two different Riemannian surfaces, for example on the Euclidean plane and on the sphere, the configuration space may be very different. Therefore, it is natural to ask what the two results above become on surfaces other than the plane. Is there a way of characterizing the curves which may be drawn ? May any smooth compact manifold be seen as the configuration space of some linkage ?
On this poster, we study three examples of surfaces : the sphere, the hyperbolic plane, and the Minkowski plane ($\mathbb{R}^2$ equipped with the quadratic form $dx^2 - dy^2$).

Definitions

Definition 1. A *linkage* $\mathcal{L}$ on a Riemannian manifold $\mathcal{N}$ is a graph (V, E) together with :
1. A function $l : E \longrightarrow \mathbb{R}^+$ (which gives the length of each edge) ;
2. A subset $F \subseteq V$ of *fixed vertices* ;
3. A function $\phi_0 : F \longrightarrow \mathcal{N}$ which indicates where the edges of F are fixed.

Definition 2. Let $\mathcal{L}$ be a linkage on a manifold $\mathcal{N}$. Let $\mathcal{M}$ be a manifold containing $\mathcal{N}$. A *realization* of a linkage $\mathcal{L}$ on $\mathcal{M}$ is a function $\phi : V \longrightarrow \mathcal{M}$ such that :
1. $\phi|_F = \phi_0$;
2. For each edge $v_1v_2 \in E$, $\delta(\phi(v_1), \phi(v_2)) = l(v_1v_2)$, where δ is the Riemannian distance on $\mathcal{M}$.

Definition 3. Let $\mathcal{L}$ be a linkage on a manifold $\mathcal{N}$. Let $W \subseteq V$. Let $\mathcal{M}$ be a manifold containing $\mathcal{N}$. The *partial configuration space of the vertices W on* $\mathcal{M}$, written $\mathcal{E}(W, \mathcal{M})$, is the following set of functions from W to $\mathcal{M}$:

$$\mathcal{E}(W, \mathcal{M}) = \{\phi|_W \mid \phi \text{ realization of } \mathcal{L}\} .$$

The differential universality theorems

Theorem 1 (Kapovich and Millson, 2002). *Let M be a smooth compact manifold. Then there exists a linkage $\mathcal{L}$ on the Euclidean plane whose configuration space is diffeomorphic to a finite disjoint union of copies of M.*

Theorem 2 (K., 2013). *Let M be a smooth compact manifold. Let $\mathcal{M}$ be the sphere, the hyperbolic plane or the Minkowski plane. Then there exists a linkage $\mathcal{L}$ on $\mathcal{M}$ whose configuration space is diffeomorphic to a finite disjoint union of copies of M.*

Algebraic subsets and Kempe's theorem

Theorem 3 (Kempe, 1875). *Let A be an algebraic curve (i. e. algebraic set) of $\mathbb{R}^2$ (the Euclidean plane) intersected with a Euclidean ball. Then there exists a linkage $\mathcal{L}$ which contains one vertex v such that the partial configuration space of the vertex v is A.*
For example, when A is a straight line, one of the solutions is the Peaucellier linkage.

Theorem 4 (Kapovich and Millson, 2002). *Let $N \geq n \geq 1$. Let $A \subseteq (\mathbb{R}^2)^n$ (where $\mathbb{R}^2$ is the Euclidean plane) be the projection of a compact algebraic set of $(\mathbb{R}^2)^N$. Then there exists a linkage $\mathcal{L}$ with vertices V and $W \subseteq V$ such that the partial configuration space of the vertices W is A.*

The notion of algebraic subset on Riemannian surfaces.
The following results indicate that the partial configuration spaces of linkages on the sphere, the hyperbolic plane or the Minkowski plane are the projections of algebraic sets. On the sphere, we define the algebraic sets as the algebraic sets of $\mathbb{R}^3$ which are contained in the Euclidean unit sphere. On the hyperbolic plane, we define the algebraic sets using the Poincaré half-plane model included in $\mathbb{R}^2$, but the use of other common models (Poincaré disk, Klein model, hyperboloid model) would give the same definition.

Theorem 5 (K., 2013). *Let $N \geq n \geq 1$. Let $\mathcal{M}$ be the sphere or the hyperbolic plane. Let $A \subseteq \mathcal{M}^n$ be the projection of a compact algebraic set of $\mathcal{M}^N$. Then there exists a linkage $\mathcal{L}$ with vertices V and $W \subseteq V$ such that the partial configuration space of the vertices W is A.*

Theorem 6 (K., 2013). *Let $N \geq n \geq 1$. We denote the Minkowski plane by $\mathbb{M}$. Let $A \subseteq \mathbb{M}^n$ be the projection of an algebraic set of $\mathbb{M}^N$ (not necessarily compact). Then there exists a linkage $\mathcal{L}$ with vertices V and $W \subseteq V$ such that the partial configuration space of the vertices W is A.*

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Gluing elementary linkages

The universality theorems give the existence of linkages whose configuration spaces satisfy the desired properties. The proof is constructive and consists in gluing some elementary linkages to create more complex ones. The Peaucellier linkage, the square linkage, the pantograph and the symmetrizer are examples of elementary linkages.

How to draw a straight line

In 1784, Watt thought of an improvement of the steam engine which required to construct a mechanical linkage with one vertex following a straight line. He only found an approximate solution, but mathematicians continued to search for an exact one. In the 1860's, Peaucellier and Lipkin finally invented a perfect straight-line linkage (Figure 1).

What happens when we consider linkages on other surfaces than the Euclidean plane ? It is an open problem whether it is possible in general to force a vertex to move on a geodesic. However, on the sphere, the hyperbolic plane and the Minkowski plane, some simple solutions exist. On the sphere, a simple vertex of length $\pi/2$ with one end fixed will draw a straight line, *i. e.* a great circle.

Although the following linkages all have the same aspect, some details must be changed specifically for each surface.

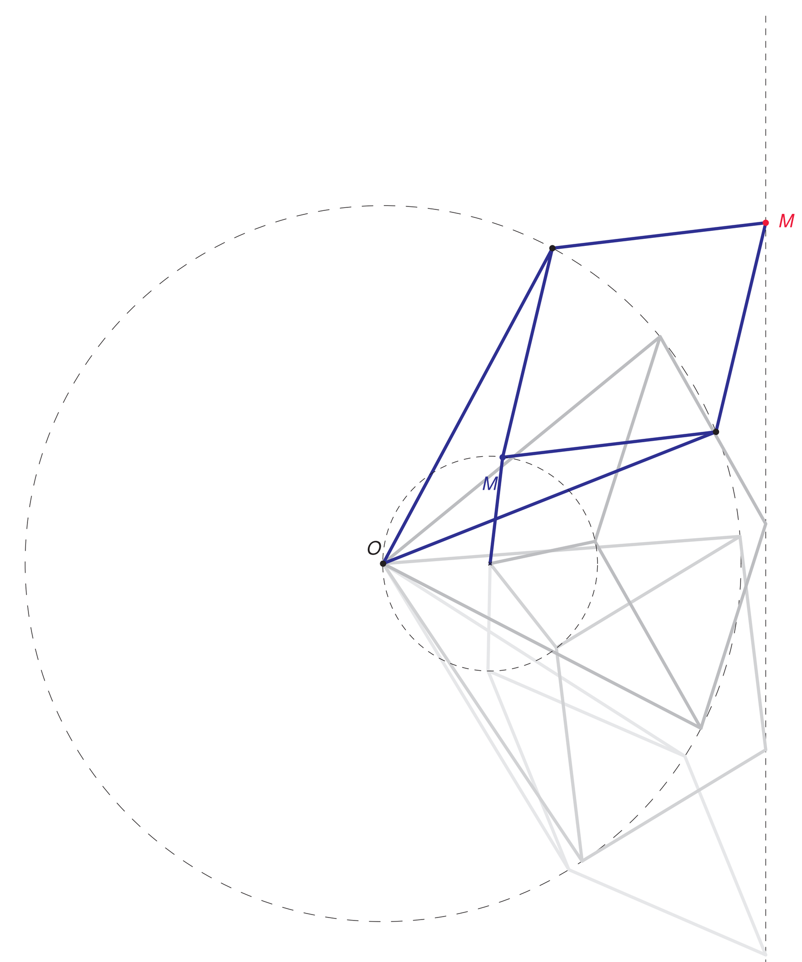


Figure 1: A straight-line linkage on the Euclidean plane. It transforms a circle into a straight line. M is the image of M' by an inversion with respect to a circle.

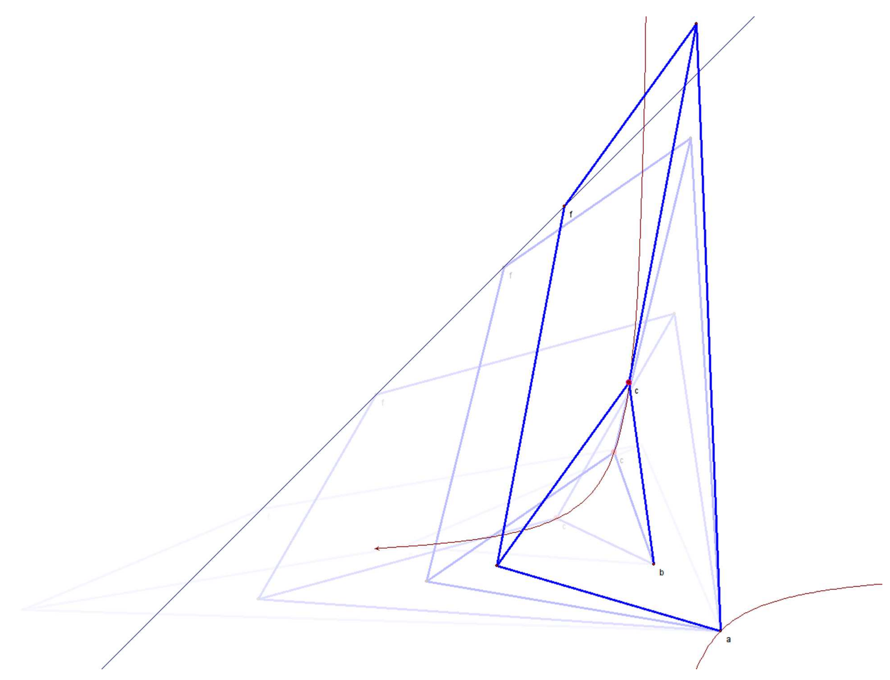


Figure 2: The Peaucellier linkage on the Minkowski plane. It transforms a hyperbola into a straight line. Three of the edges are "timelike" while four of them are "spacelike". This linkage is not able to draw "lightlike" lines.

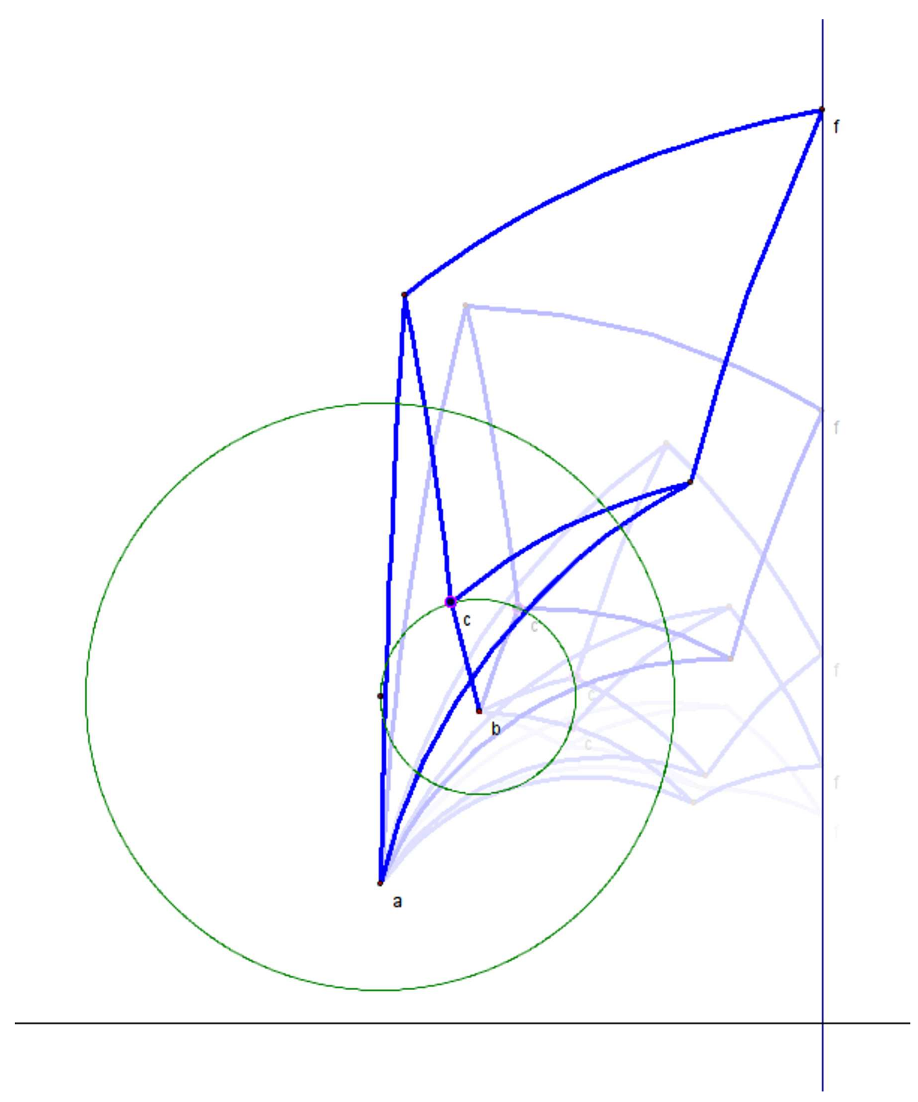


Figure 3: A straight-line linkage on the Poincaré half-plane model of the hyperbolic plane. It transforms a circle into a straight line : it is able to draw any geodesic or horocycle. In fact, f is the image of c by an inversion with respect to a circle.

The symmetrizer

Besides the Peaucellier linkage, several elementary linkages have a geometric role. On the sphere, the symmetrizer (Figure 4) realizes symmetry with respect to a vectorial plane. The vertex B is symmetric to E with respect to the plane $G^{\perp}$... except in the case of degenerate configurations, which we want to avoid.

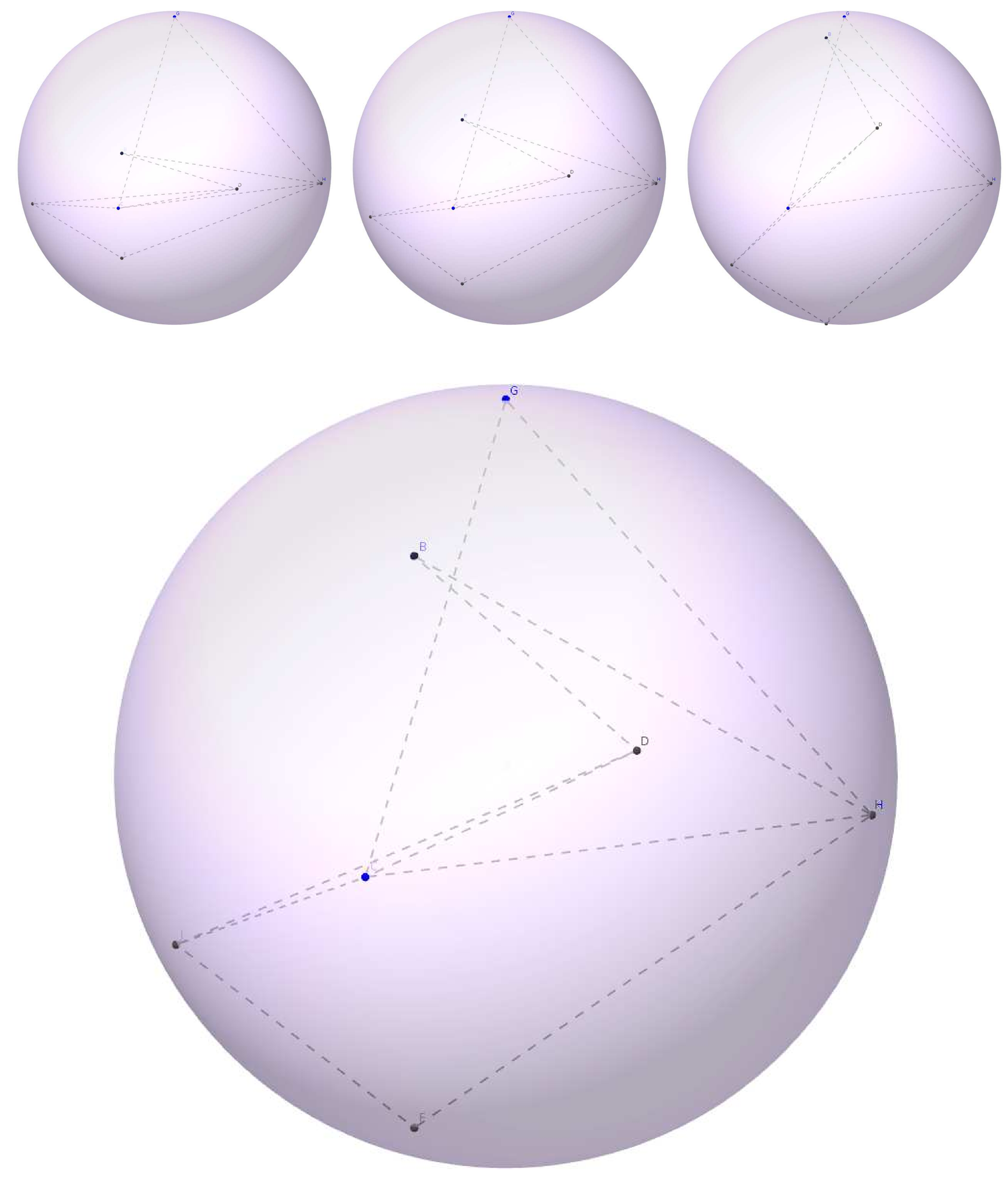
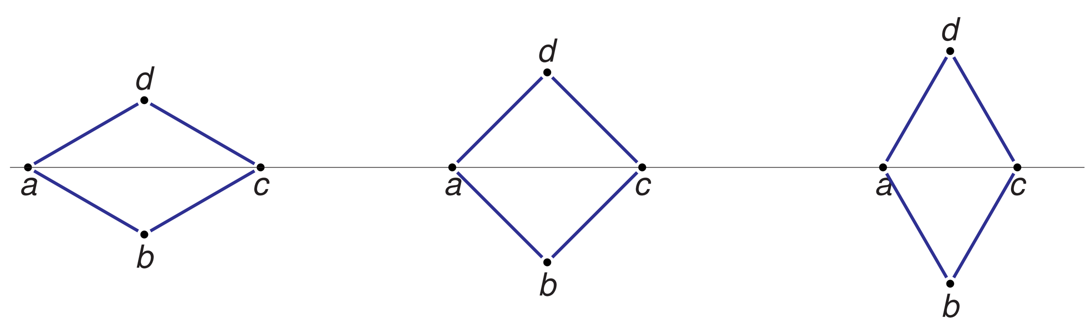
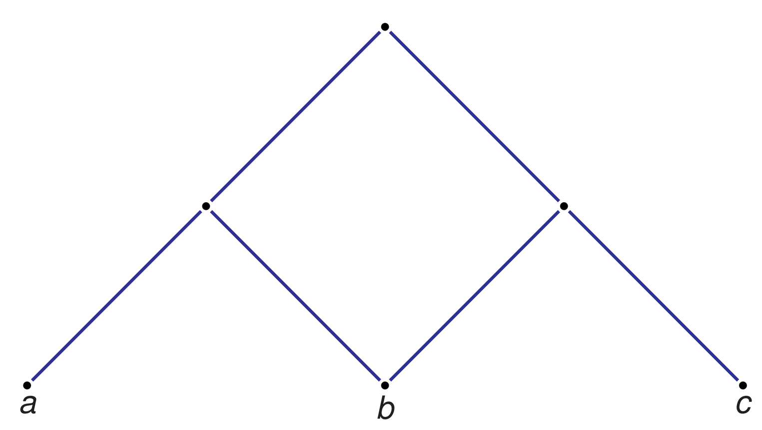


Figure 4: The symmetrizer on the sphere.

On the Euclidean plane, the symmetry may be realized using a square *abcd* with two vertices *a* and *c* forced to move on a straight line by a Peaucellier linkage. The vertices *d* and *b* are symmetric with respect to the line. Note that the square *abcd* has to be rigidified (see *The configuration space of the square linkage* below).



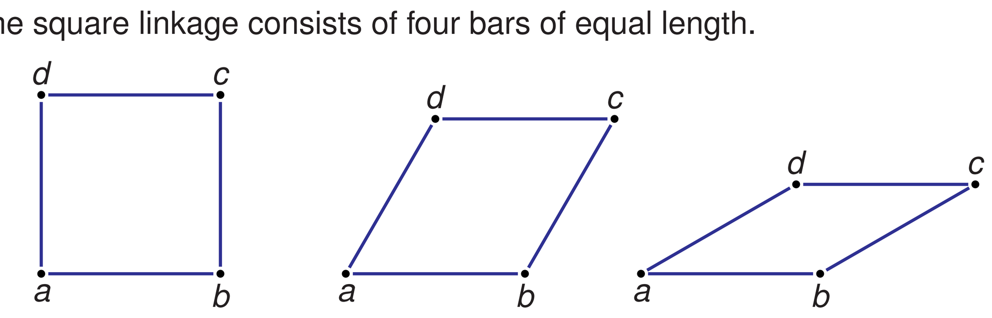
The pantograph



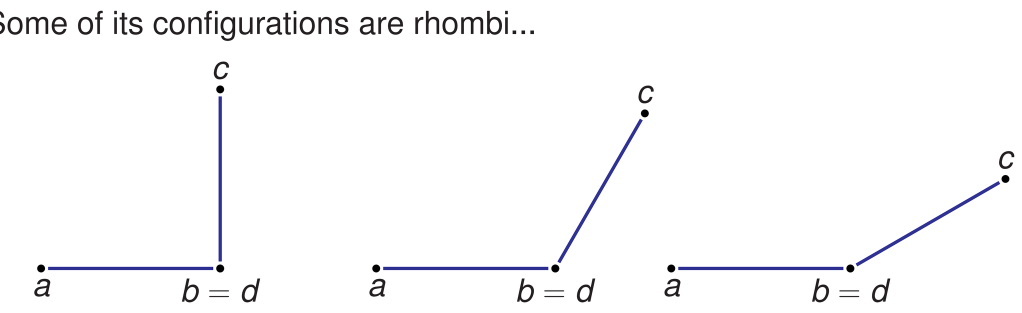
The pantograph is a well-known linkage on the Euclidean plane. If the vertex *a* is pinned down to the origin of the plane, then the vertex *b* is the image of the vertex *c* by a homothety of ratio $1/2$.
This linkage has no obvious equivalent on the sphere or on the hyperbolic plane, because there is no natural notion of homothety on these manifolds.

The configuration space of the square linkage

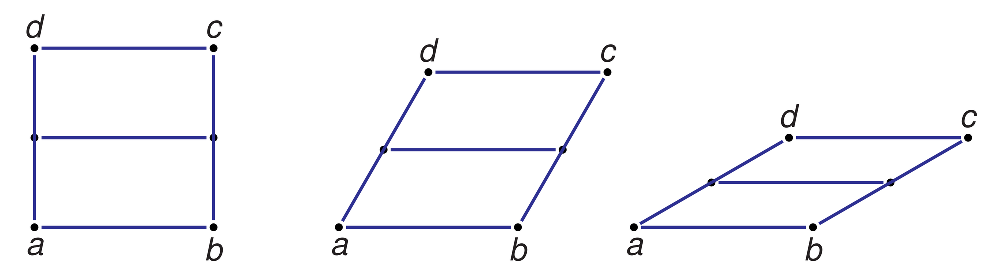
The square linkage consists of four bars of equal length.



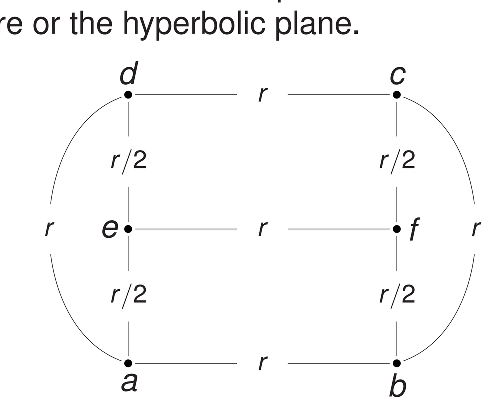
Some of its configurations are rhombi...



... but there are also degenerate configurations in which two vertices coincide. To avoid the degenerate configurations, we brace the square the following way :



This technique works on the Euclidean plane and on the Minkowski plane, but not on the sphere or the hyperbolic plane.



The rigidified square linkage, with one fixed vertex *a*, has a configuration space diffeomorphic to $\mathcal{S}^1 \times \mathcal{S}^1$ on the Euclidean plane and the Minkowski plane, but it is diffeomorphic to $\mathcal{S}^1$ on the sphere and the hyperbolic plane.