Superheavy subsets and noncontractible Hamiltonian circle actions

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1. The problem of non-displaceability

The problem of non-displaceability is one of important problems in symplectic topology

Denition 1

(M,ω): symplectic manifold

A subset X of M is non-displaceable by symplectomorphisms (Hamiltonian diffeomorphisms) if there exists no symplectomorphism (Hamitonian diffeomorphism) f such that $\overline{X} \cap f(X) = \phi$.

To solve the above problem, M. Entov and L. Polterovich defined the heaviness and the superheaviness of closed subsets in closed symplectic manifolds.

We omit the definiton of them, but we note that they are defined in the term of the Hamiltonian Floer theory.

Theorem 2 (Entov-Polterovich, [EP09])

・A heavy subset is non-displaceable by Hamiltonian diffeomorphisms.

・A superheavy subset is non-displaceable by symplectomorphisms.

Remark: In this poster, we denote ``superheavy subsets with respect to the foundamental class'' by superheavy subsets. This is not usual notation.

History:

Entov-Polterovich(EP03): Definiton and construction of Calabi quasi-morphisms on the group of Hamiltonian diffeomorphisms.

Biran-Entov-Polterovich(BEP): First application of Calabi quasi-morphisms to non-displaceability

Entov-Polterovich(EP09):
Definition of heaviness and superheaviness

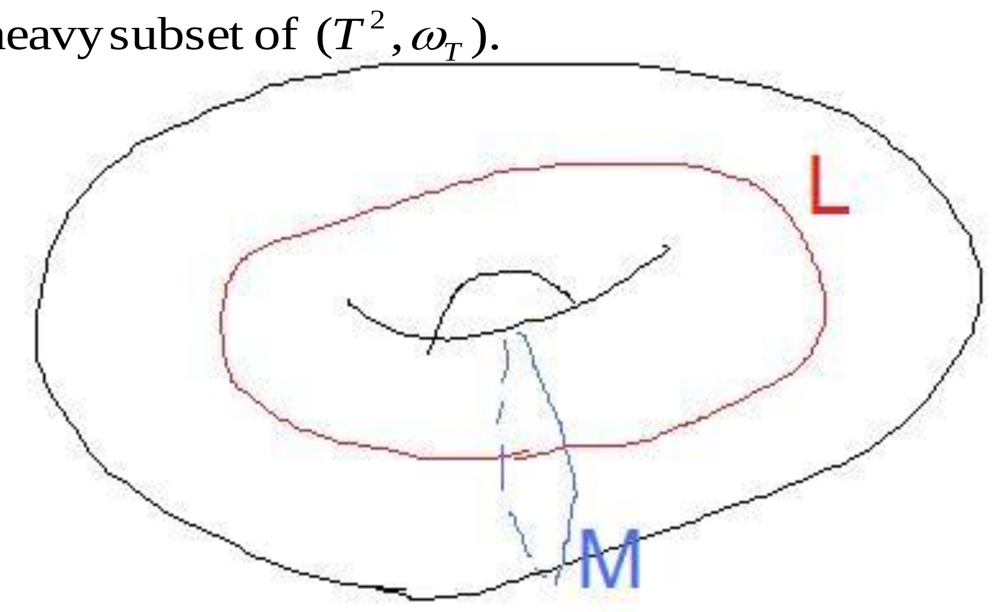
2. Our Result

Theorem 3

(T^2, ω_T): the 2-torus with coordinate (p,q) and the symplectic form $dp \wedge dq$

M, L: the meridean, longitude curve of T^2, respectively

The union $M \cup L$ is a superheavy subset of (T^2, ω_T).



The above subset is trivially non-displaceable by homeomorphisms.

But we can obtain a non-trivial result of non-displaceability.

Corollary 4

(CP^n, ω_{FS}): the complex projective space with the Fubini-Study form

$C := \{[z_0, \cdots, z_n] \mid |z_0| = \cdots = |z_n|\} \subset CP^n$ (the Clifford torus)

Then a subset $C \times (M \cup L)$ of $CP^n \times T^2$ is non-displaceable by symplectomorphisms.

Proof of Corollary 4

The Clifford torus C is known to be a superheavy subset of (CP^n, ω_{FS}).

A product of superheavy subsets is known to be also superheavy (EP09).

By Theorem 3, $C \times (M \cup L)$ is a product of superheavy subsets and hence non-displaceable by symplectomorphisms.

3. Proof of Theorem 3

The following proposition is the important idea in the proof of Theorem 3.
The proof of this proposition is based on the idea of K. Irie (I).

Proposition 5

(M,ω): rational symplectic manifold

$\alpha \neq 0 \in [S^1, M]$, U: open subset of M,

H: Hamiltonian function on M (i.e. $H \in C^\infty(M)$)

Assumption:

(1) $\phi_H^{\ 1}|_U = id$

(2) For any $x \in U$, $\gamma^x := (t \mapsto \phi_H^{\ t}(x)) = \alpha \in [S^1, M]$

(3) $\alpha \notin [S^1, U]$

Then U is ``strongly null''.

To prove Theorem 3 by using Proposition 5, we use the following proposition.

Proposition 6 (essentially Entov-Polterovich, EP09)

$F_1, \cdots, F_k$: functions on M which satisfies that $\{F_i, F_j\} = 0$ for any i, j.

$\Phi := (F_1, \cdots, F_k): M \to R^k$. Fix $p \in R^k$.

Assume that for any $q \neq p$, $\Phi^{-1}(q)$ is strongly null.

Then $\Phi^{-1}(p)$ is superheavy subset of (M,ω).

Proof of Theorem 3

Take $\Phi: M \to R$ such that $\Phi(x) = 0 \Leftrightarrow x \in M \cup L$.

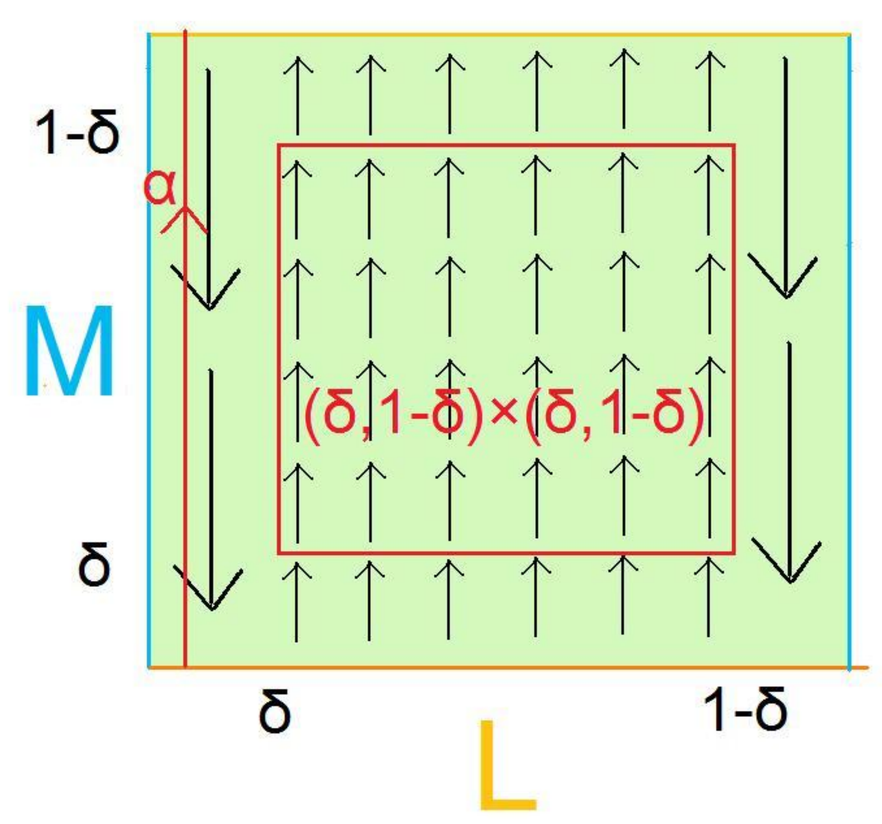
$\forall \varepsilon \neq 0, \exists \delta > 0$ such that $\Phi^{-1}(\varepsilon) \subset U_\delta := (\delta, 1-\delta) \times (\delta, 1-\delta) \subset T^2$.

Consider a Hamiltonian function H such that $H(p,q) = p$ for $p \in [\delta, 1-\delta]$.

Define α by $[t \mapsto (0,t)] \in [S^1, T^2]$.

Then α, U_δ and H satisfies the conditions of Proposition 5.

Thus by Proposition 6, $M \cup L = \Phi^{-1}(0)$ is a superheavy subset of (T^2, ω_T).



4. Reference

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