

# Maximal dimension totally geodesic foliations on the complex hyperbolic space

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## 1. Totally geodesic foliations of the real hyperbolic space

Totally geodesic foliations of the real hyperbolic space  $\mathbb{H}^n$  in codimension 1 are well understood. Every complete totally geodesic codimension 1 submanifold of  $\mathbb{H}^n$  is an isometric copy of  $\mathbb{H}^{n-1}$ . Classical **geometric** classification of foliations with such leaves is as follows

### Theorem

Every codimension 1 totally geodesic  $\mathcal{C}^2$  foliation of  $\mathbb{H}^n$  is orthogonal to a curve (defined on  $\mathbb{R}$ ) of geodesic curvature  $\leq 1$ .

D. Ferus, On isometric immersions between hyperbolic spaces, Math. Ann. 205 (1973), 193–200.

Looking at the ideal boundary of leaves as well as the hyperbolic space the **boundary** classification could be provided

### Theorem

The set of codimension 1 totally geodesic  $\mathcal{C}^k$  foliations of  $\mathbb{H}^n$  (with horocycle foliations excluded) modulo action of the group  $O(n-1) \times \mathbb{R} \times \mathbb{Z}_2$  is in one-to-one correspondence with the set of all  $\mathcal{C}^{k-1}$  functions  $z : [0, \pi] \rightarrow \mathbb{S}^{n-1}$  such that  $|z'| \leq 1$  in some natural parametrization.

K. B. Lee, S. Yi, Metric foliations on hyperbolic spaces, J. Korean Math. Soc. 48(1) (2011), 63–82.

In the hyperboloid model every totally geodesic hypersurface is represented by a time-like hyperplane in the Lorentz space  $(\mathbb{R}^{1,n}, \langle \cdot, \cdot \rangle)$ . Thus its Lorentz normal vector is a point in de Sitter space  $\Lambda^n$  consisting of all unit space-like vectors in  $\mathbb{R}^{1,n}$ . Looking from the **conformal** viewpoint

### Real Theorem

Totally geodesic codimension 1  $\mathcal{C}^1$  foliations of  $\mathbb{H}^n$  are in one-to-one correspondence with the set of unbounded curves  $\Gamma : \mathbb{R} \rightarrow \Lambda^n$  satisfying the following condition

$$\langle \Gamma(\tau_1) | \Gamma(\tau_2) \rangle \geq 1 \quad \text{for all } \tau_1, \tau_2 \in \mathbb{R}.$$

In the class  $\mathcal{C}^2$  this condition means simply that the curve  $\Gamma$  is time-or-light-like (and future oriented).

M. Czarnecki, R. Langevin, Totally umbilical foliations on hyperbolic spaces, in preparation.

For more wide explanation see

M. Badura, M. Czarnecki, *Recent progress in geometric foliation theory* in *Foliations 2012*, World Scientific 2013, 9–21.

M. Czarnecki, P. Walczak, *Extrinsic geometry of foliations* in *Foliations 2005*, World Scientific 2006, 149–167.

and the bibliography therein.

## 2. The complex hyperbolic space and the complex de Sitter space

**Definition** Consider the Hermitian form in  $\mathbb{C}^{1,n}$  given by

$$\langle\langle z | w \rangle\rangle = -z_0 \bar{w}_0 + z_1 \bar{w}_1 + \dots + z_n \bar{w}_n.$$

The complex hyperbolic  $n$ -space  $\mathbb{C}\mathbb{H}^n$  is the (complex) projectivization of negative vectors in  $\mathbb{C}^{1,n}$  with respect to  $\langle\langle \cdot | \cdot \rangle\rangle$  while the complex de Sitter space  $\mathbb{C}\Lambda^n$  is the projectivization of positive vectors.

### Proposition

Every totally geodesic submanifold of the complex hyperbolic space is totally complex or totally real.

In particular, minimal (real) codimension of totally geodesic submanifold is 2 and every complete such a submanifold of  $\mathbb{C}\mathbb{H}^n$  is an isometric copy of  $\mathbb{C}\mathbb{H}^{n-1}$ .

W. Goldman, Complex Hyperbolic Geometry, Oxford University Press 1999.

Every totally geodesic codimension 2 submanifold of  $\mathbb{C}\mathbb{H}^n$  is the projectivization of complex hyperplane which is complex-time-like. Thus it is represented by a positive vector i.e. belonging to  $\mathbb{C}\Lambda^n$ .

## 3. Codimension 2 totally geodesic foliations of the complex hyperbolic space

Using conformal approach very similar to the real case we have

### Complex Theorem

Totally geodesic codimension 2 foliations of  $\mathbb{C}\mathbb{H}^n$  are in one-to-one correspondence with the set of unbounded curves  $\Gamma : \mathbb{C} \rightarrow \mathbb{C}\Lambda^n$  which are complex-time-or-light-like.

Although the normalized vector field orthogonal to such foliation is constant along every leaf it is not necessary integrable. Probably some weaker assumptions than holomorphicity guarantee this integrability.

Then we could formulate more geometrical condition

### Corollary

If the field of vectors orthogonal to a codimension 2 totally geodesic foliation of  $\mathbb{C}\mathbb{H}^n$  is integrable then this foliation is orthogonal to a 2-dimensional Hadamard manifold.

### Problem

Apply this method to minimal codimension (equal 4) totally geodesic foliations of the quaternionic hyperbolic space.