Index Theory for Riemannian Foliations

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The problem

Given a simple foliation i. e. a smooth submersion $f : M \to N$ between closed manifolds, we can consider two kinds of elliptic geometric operators: those acting on the base N alone, and those which act only along the fibers of f. The former, the "transversal" case, is by now classical, culminating in the Atiyah-Singer Index Theorem and its local version due to Getzler, while the latter, the "leafwise" or "longitudinal" case involves the study of elliptic families parametrized by the base, thus leading to the Atiyah-Singer Families Index Theorem and Bismut's local version of it. For general Riemannian foliations $(M, \mathcal{F}, g)$ we can extend these ideas to obtain transversally and leafwise elliptic differential operators, and ask the corresponding questions. The Longitudinal Index Theorem of Connes and Skandalis gives one satisfying answer while the transversal situation is not so well understood yet, one obvious reason being the disappearence of a nice base space.

We present here an index theorem for a large class of transversally elliptic operators acting on the basic sections of a foliated bundle. After setting up the operator theoretic apparatus, we use Molino theory to replace our operator by an elliptic differential operator acting on the regular part of a Whitney stratified space W. We desingularize this space and can, by a careful heat kernel analysis, express the index of the original operator in terms of Atiyah-Singer type convergent integrals over certain submanifolds of the desingularization which cover the strata of W. In addition, there may appear η-invariants, as introduced by Atiyah, Patodi, and Singer [APS], which are not easily explained.

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The setup

Consider a foliation $(M, \mathcal{F}, g)$ where (M, g) is a closed, m-dimensional Riemannian manifold, $\mathcal{F}$ is a foliation with codimension q and leaves of dimension $p = m - q$. We denote the tangent space to the leave through x by $T_{V,x}$ and its g-orthogonal complement in T_xM by $T_{H,x}$; accordingly, the metric is split as $g = g_H \oplus g_V$. In analogy with the case of a simple foliation, we require that the Lie derivative $\mathcal{L}_X g_H$ vanishes for any $X \in T_V$. Next we consider a smooth hermitian vector bundle (E, h), of rank k, with metric covariant derivative ∇^E over M. We assume that E is associated to a principal G-bundle $\pi : P_G \to M$ which we assume to be **foliated**, in the sense that $\mathcal{F}$ lifts to a G-invariant foliation $\mathcal{F}_P$ on P_G which is transversal to the fibers of π, cf. [Mo, p.54]. We require ∇^E to be **basic**, i. e. its connection one-form ω is basic with respect to $\mathcal{F}_P$. Then we call a smooth section, s, of E basic if $\nabla^E_X s = 0$ for $X \in T_V$; the space of smooth basic sections will be denoted by $C^\infty_b(M, E)$. It generates a Hilbert space, $L^2_b(M, E)$, suitable for the relevant spectral theory.

Next we introduce a smooth differential operator, D, of order l on E which maps $C^\infty_b(M, E)$ to $L^2_b(M, E)$ and is symmetric. We assume that D is **transversally elliptic** i. e. that the principal symbol $\hat{D}(\xi)$ is invertible for all $\xi \in T^*_H M$. These operators are Fredholm if $\mathcal{F}$ is generated by a compact Lie group, G, acting on M, (see [A, Lecture 2] and below). For explicit index formulae, we restrict attention to the important class of **transversal Dirac operators** acting on the basic sections of **basic Clifford bundles** (E, h, ∇^E) over M; cf. [BKR1, 2.3, 2.4] for the details. They also have to satisfy a technical condition holding in most interesting situations, see [BKR1, Sec.6].

Reduction to an equivariant operator and index calculation

We unitarily identify the operator D^E_b, defined before, with a G-equivariant operator $D^{\widehat{F}}_{\mathrm{inv}}$, restricted to the G-invariant sections of a vector bundle $\widehat{F} \to \widehat{W}$ that will be constructed; the full details are given in [BKR1, Sec. 3.1].

$$\begin{tikzcd}
 & \pi^*(E,h,\nabla^E) \arrow[r] \arrow[d] & (\widehat{M},\widehat{\mathcal{F}},\hat{g}) \arrow[r, "\hat{\pi}"] \arrow[d, "\pi"] & \widehat{W} \arrow[d] & \widehat{F} \arrow[l] & D^{\widehat{F}}_{\mathrm{inv}} \\
D^E_b & (E,h,\nabla^E) \arrow[r] & (M,\mathcal{F},g) \arrow[r, "\hat{\pi}/G"] & W = \widehat{W}/G & F \arrow[l] &
\end{tikzcd}$$

We introduce first the principal $G = O(q) \times U(k)$- bundle $\widehat{M}$ of orthonormal frames for $T_H M \otimes E$, with projection $\pi : \widehat{M} \to M$, lift the foliation to $\hat{\mathcal{F}}$, and pull back (E, h, ∇^E) and g to $\hat{M}$. Then π^*E is foliated and ∇^{π^*E} is basic. Moreover, G acts on π^*E preserving the basic $\widehat{\mathcal{F}}$-sections. Next we observe that $\widehat{M}$ is transversally parallelizable (cf. [Mo, p.83]), such that Molino's Structure Theorem [loc. cit. Thm. 4.2] is applicable. Hence there is a fibration $\hat{\pi} : \widehat{M} \to \widehat{W}$ where $\widehat{W}$ is the *basic manifold* for $(M, \mathcal{F})$, and the fibers of $\hat{\pi}$ are the leaf closures of $\widehat{\mathcal{F}}$. $\widehat{W}$ is also a G-space, and dividing out the G-action gives a continuous surjection $\hat{\pi}/G : M \to W$, with fibers the leave closures of $\mathcal{F}$. As orbit space of a compact Lie group, W is Whitney stratified and hence induces a Whitney stratification of $\mathcal{F}$; this is the tool on which the index calculation is based. Now we define a smooth hermitian vector bundle $\widehat{F}$ over $\widehat{W}$ by defining $\widehat{F}_{\hat{w}}$:= vector space of $(M, \mathcal{F})$-basic sections of π^*E, restricted to the leave closure $\hat{\pi}^{-1}(\hat{w})$. Then one can show that the operator D^E_b is unitarily equivalent to a G-equivariant Dirac-type operator $D^{\widehat{F}}_{\mathrm{inv}}$, restricted to the G-invariant sections of $\widehat{F}$. Hence we are reduced to computing the index of $D^{\widehat{F}}_{\mathrm{inv}}$, or rather its reduction, $D^{\widehat{F},+}_{\mathrm{inv}}$ by an anticommuting isometry of $\widehat{F}$.

The index calculation

Using the arguments given in [BH, Sect.1], we construct a hermitian vector bundle F' over W_{reg}, the regular part or the top stratum of W, and a differential operator, $D^{F'}$, on $C^\infty_c(W_{\mathrm{reg}}, F')$ which is unitarily equivalent to $D^{\widehat{F}}_{\mathrm{inv}}$. The bundle F' is defined by $F'_{Gp} := C^\infty(Gp, E|Gp)^G \simeq E_p^{G_p}$, and the unitary isomorphism is given by $s \mapsto s|Gp$. $D^{F'}$ turns out to be essentially self-adjoint in the corresponding L^2-space, such that we are left with computing the index of $D^{F',+}$. This is done, essentially, by constructing the desingularization of W and observing that the heat kernel of $(D^{F'})^2$ defines an "asymptotic density" on each stratum which allows the computation of its contribution to the index formula. In fact, the desingularization is, roughly speaking, constructed by cutting out tubular neighbourhoods of minimal strata and doubling the complement which is a less complicated Whitney stratified space, whose contribution can be treated inductively. Hence the main contributions in each step will come from the horizontal part, the stratum, and the vertical part, a sphere, of the tube which contribute the Atiyah-Singer type contribution horizontally and the η-type contribution vertically. They are explicit in each specific example but involve the modified geometry of the desingularization, instead of the geometry of the original manifold. A full treatment of this analysis is in preparation [BKR2].

Gorokhovsky and Lott [GL] have produced a different formula in a more special situation; progress might result from understanding the relations between the two formulas.

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