Entropy-like invariants for groups pseudogroups and foliations

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Introduction

In classical dynamical systems one of the most fundamental invariants of a continuous map $f : X \to X$ is its topological entropy $h_{top}(f)$ which measure the complexity of the system. When the entropy is positive, it reflects some chaotic behavior of the map f.
In foliation theory, any so called nice covering $\mathcal{U}$ of a compact foliated manifold (M, F) determines a finitely generated holonomy pseudogroup $(H_{\mathcal{U}}, H_{\mathcal{U}1})$ generated by a finite generating set $H_{\mathcal{U}1}$. Also, there exists a corresponding notion of a topological entropy for a group action or pseudogroup action on a compact metric space. For any foliated manifold (M, F), the action of a holonomy pseudogroup on a complete transversal contains complete information about the dynamics of (M, F). It does not depend on the choice of the transversal up to an equivalence of pseudogroups. Therefore, a foliated manifold can be considered as a generalized dynamical system.

In classical theory of dynamical systems a continuous map $f : X \to X$ determines an f-invariant measure μ and one can define a measure entropy $h_\mu(f)$ with respect to μ. The important relation between topological entropy and measure entropy of a map $f : X \to X$ is established by the Variational Principle, which asserts that

$$h_{\text{top}}(f) = \sup\{h_\mu(f) : \mu \in M(X, f)\}$$

i.e. topological entropy equals to the supremum $h_\mu(f)$, where μ ranges over the set $M(X, f)$ of all f-invariant Borel probability measures on X.

In classical dynamical systems there are relations between the topological entropy of a continuous map $f : X \to X$ and Hausdorff dimension. More than thirty years ago Bowen [4] provided a definifion of topological entropy of a map which resembles the definition of Hausdorff dimension. A dimensional type approach to topological entropy of a single continuous map one can find for example in [1], [10] or [7]. A cyclic group or semigroup $< f >$ generated by a single map $f : X \to X$ has linear growth. Therefore it is difficult to adopt ideas and techniques presented for groups of linear growth to finitely generated groups or pseudogroups which growth is rarely linear. The goal of the talk is to present interrelations between dimension theory and the theory of generalized dynamical systems.

Topological entropy of a pseudogroup and local measure entropy

In [6] Ghys, Langevin and Walczak defined the topological entropy of a finitely generated pseudogroup and introduced a notion of a geometric entropy of a foliation. The problem of defining good measure theoretical entropy for foliated manifolds which would provide an analogue of the variational principle for geometric entropy of foliations is still open. In general, there are many examples of foliations that do not admit any non-trivial invariant measure. Even in a case when an invariant measure exists, it is not clear how to define its measure-theoretic entropy.
We generalize the notion of local measure entropy introduced by Brin and Katok [5] for a single map $f : X \to X$ to a finitely generated pseudogroup (G, G_1) acting on a metric space X. We define a local upper measure entropy $h^G_\mu(x)$ and a local lower measure entropy $h_{\mu,G}(x)$ of (G, G_1) at a point $x \in X$ with respect to a Borel probability measure μ on X.
The main result of [2] is an analogue of the partial Variational Principle for pseudogroups which reads as follows:

Theorem 1.

Let (G, G_1) be a finitely generated group of homeomorphisms of a compact closed and oriented manifold M. Let E is a Borel subset of M, $s > 0$ and μ_{vol} the natural volume measure on M.
If the local measure entropy $h^G_{\mu_{\text{vol}}}(x) \leq s$, for all $x \in E$, then the topological entropy $h_{\text{top}}((G, G_1), E) \leq s$.

Theorem 2.

Let (G, G_1) be a finitely generated pseudogroup on a compact metric space X. Let E is a Borel subset of X and $s > 0$. Denote by μ a Borel probability measure on X.
If the local measure entropy $h_{\mu,G}(x) \geq s$, for all $x \in E$, and $\mu(E) > 0$ then the topological entropy $h_{\text{top}}((G, G_1), E) \leq s$.

Next, we introduce a special class class of measures on X, called $G-$ homogeneous measures.

Theorem 3.

If a finitely generated pseudogroup (G, G_1) acting on a compact metric space (X, d) admits a G-homogeneous measure then the local measure entropy $h^G_\mu(x)$ is constant and it does not depend on the point $x \in X$. Moreover:
For a finitely generated pseudogroup (G, G_1) acting on a compact metric space X and admitting a G-homogeneous measure μ on X we have

$$h_{\text{top}}(G, G_1) = h^G_\mu,$$

where h^G_μ is the common value of local measure entropies $h^G_\mu(x)$.

New entropy-like invariants

Here we present and apply the theory of Carathéodory structures (or C-structures), studied by Pesin ([8], [9]) and Pesin and Pitskel ([10]), which are the powerful generalization of the classical construction of Hausdorff measure. Pesin introduced a C-structure axiomatically by describing its elements and relation between them.
A Carathéodory structure τ defined on a metric space X determines the Carathéodory dimension $\dim_{C,\tau}(Z)$ of a subset $Z \subset X$. Another procedure leads to definition of two other basic characteristics of dimensional type: the lower and upper capacity of a set $Z \subset X$. The main results of [3] are as follows:

Theorem 4.

For a finitely generated pseudogroup (G, G_1) there exists a C- structure with upper capacity that coincides with the topological entropy of (G, G_1).

We denote by E a class of continuous and decreasing functions $f : [0, \infty) \to [0, \infty)$ with $\lim_{x\to\infty} f(x) = 0$. Now, we fix a pseudogroup (H, H_1) acting on a metric space X. Any function $f \in E$ and the pseudogroup (H, H_1) determine a class of C-structures $\Gamma(f)_\delta = \{(F_\delta, \xi, \eta, \psi) : \delta > 0\}$ and the limit C-structure $\Gamma(f)$ on X. The upper capacity of a set $Z \subset X$, with respect the limit C-structure $\Gamma(f)$, is denoted here by $\overline{CP(f)}_Z$.

We apply the Theorem 4 to get some estimations of the geometric entropy $h_{\text{geom}}(F, g)$ of a compact foliated manifold (M, F), which describes the global dynamics of (M, F). It is known that a compact foliated manifold (M, F) with fixed so called nice covering U determines a finitely generated holonomy pseudogroup $(H(U), H_1(U))$ acting on the transversal T_U. Here, the finite generating set $H_1(U)$ consists of elementary holonomy maps corresponding to overlapping charts of U.

Theorem 5.

Given a finitely generated pseudogroup (H, H_1) acting on a compact metric space X. Assume that for $f, g \in E$ and for any $x \in [0, \infty)$ the inequalities $f(x) \leq e^{-x} \leq g(x)$ hold. Then, for any subset $Z \subset X$ we get

$$\overline{CP(f)}_Z \leq h_{\text{top}}((H, H_1), Z) \leq \overline{CP(g)}_Z.$$

As a corollary we get two classes of dimensional type estimations of the geometric entropy of foliations.

Theorem 6.

Assume that for $f_1, f_2 \in E$ and for any $x \in [0, \infty)$ the inequalities $f_1(x) \leq e^{-x} \leq f_2(x)$ hold. For any nice covering U of a compact foliated manifold(M, F) endowed with a Riemannian structure g, denote by $\text{diam}(U)$ the maximum of the diameters of the plaques of U measured with respect to the Riemannian structures induced on the leaves. Then

$$h^{\text{lower}}_{\text{geom}}(F, f_1) \leq h_{\text{geom}}(F, g) \leq h^{\text{upper}}_{\text{geom}}(F, f_2),$$

where

$$h^{\text{lower}}_{\text{geom}}(F, f_1) = \sup\left\{\frac{1}{\text{diam}(U)}\overline{CP(f_1)(H(U), H_1(U))}_{T_U} : U\text{-nice cover of } M\right\},$$

$$h^{\text{upper}}_{\text{geom}}(F, f_2) = \sup\left\{\frac{1}{\text{diam}(U)}\overline{CP(f_2)(H(U), H_1(U))}_{T_U} : U\text{-nice cover of } M\right\}.$$

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