Geometry and Foliations 2013
Komaba, Tokyo, Japan

Non-wandering, recurrence, p.a.p. and *R*-closed properties for flows and foliations

Tomoo YOKOYAMA

1. Introduction

In the recent papers [6]-[11], we study pointwise almost periodic (p.a.p), recurrent, non-wandering, and R-closed properties for flows. By a flow, we mean a continuous action of a topological group G on X. Also we define these notions for decompositions (in particular foliations). Using them, we study codimension one and two foliations. By a decomposition, we mean a family $\mathcal{F}$ of pairwise disjoint nonempty subsets of a set X such that $X = \sqcup\mathcal{F}$. In this talk, we survey these results. The following relations for decompositions of compact Hausdorff spaces hold [6][10]:

$$R\text{-closed} \Rightarrow \text{pointwise almost periodic} \Rightarrow \text{recurrent} \Rightarrow \text{non-wandering}.$$

Let $\mathcal{F}$ be a decomposition of a topological space X. An element L of $\mathcal{F}$ is said to be recurrent if either it is compact or $\overline{L} - L$ is not closed. An element L of $\mathcal{F}$ is non-wandering if it is contained in the closure of the union of recurrent elements. A decomposition $\mathcal{F}$ is said to be recurrent (resp. non-wandering) if so is each element of $\mathcal{F}$. We call that $\mathcal{F}$ is pointwise almost periodic (p.a.p.) if the set of all closures of elements of $\mathcal{F}$ also is a decomposition. Then denote by $\hat{\mathcal{F}}$ the decomposition of closures and by $M/\hat{\mathcal{F}}$ the quotient space, called the orbit class space. For any $x \in X$, denote by L_x the element of $\mathcal{F}$ containing x. $\mathcal{F}$ is R-closed if $R := \{(x, y) \mid y \in \overline{L_x}\}$ is closed. For a subset $A \subseteq X$, A is saturated if $A = \sqcup_{x \in A} L_x$. A decomposition $\mathcal{F}$ is upper semicontinuous (usc) if each element of $\mathcal{F}$ is both closed and compact and, for any $L \in \mathcal{F}$ and for any open neighborhood U of L, there is a saturated neighborhood of L contained in U. The following relations for p.a.p. decompositions of compact metrizable spaces hold[9]:

$$\mathcal{F} : R\text{-closed} \Longleftrightarrow \hat{\mathcal{F}} : R\text{-closed} \Longleftrightarrow \hat{\mathcal{F}} : \text{usc} \Longleftrightarrow M/\hat{\mathcal{F}} : \text{Hausdorff}$$

By a continuum we mean a compact connected metrizable space. A continuum $A \subset X$ is said to be annular if it has a neighbourhood $U \subset X$ homeomorphic to an open annulus such that $U - A$ has exactly two components each of which is homeomorphic to an annulus. We call that

Partly supported by the JST CREST Program at Department of Mathematics, Hokkaido University.

a subset $C \subset X$ is a circloid if it is an annular continuum and does not contain any strictly smaller annular continuum as a subset. We say that a minimal set $\mathcal{M}$ on a surface homeomorphism $f : S \to S$ is an extension of a Cantor set if there are a surface homeomorphism $\widetilde{f} : S \to S$ and a surjective continuous map $p : S \to S$ which is homotopic to the identity such that $p \circ f = \widetilde{f} \circ p$ and $p(\mathcal{S})$ is a Cantor set which is a minimal set of $\widetilde{f}$.

In [5], it has shown that a weakly almost-periodic orientation-preserving homeomorphism on $\mathbb{S}^2$ which is not periodic has exactly two fixed points and the closure of each regular orbit is annular continuum. From now on, let f an R-closed homeomorphism on a connected orientable closed surface S. Now we state the results for R-closed homeomorphisms.

Proposition 1.1. *[7] If $S = \mathbb{S}^2$ and f is not periodic but orientation-preserving (resp. reversing), then the minimal sets of f (resp. f^2) are exactly two fixed points and other circloids and $\mathbb{S}^2/\hat{f} \cong [0, 1]$.*

Theorem 1.2. *[7] If S has genus more than one, then each minimal set of f is either a periodic orbit or an extension of a Cantor set.*

In [3], it has shown that an invariant continuum K of a non-wandering homeomorphism of a compact orientable surface satisfies one of the following holds: (1) f has a periodic point in K; (2) K is annular; (3) $K = S = \mathbb{T}^2$. Moreover, it has shown [2] that a minimal set $\mathcal{M} \neq \mathbb{T}^2$ of a non-wandering toral homeomorphism satisfies one of the following holds: (1) $\mathcal{M}$ is a periodic orbit; (2) $\mathcal{M}$ is the orbit of a periodic circloid; (3) $\mathcal{M}$ is the extension of a Cantor set.

Theorem 1.3. *[7] If $S = \mathbb{T}^2$ and f is neither minimal nor periodic, then either each minimal set of f is a finite disjoint union of essential circloids or there is a minimal set which is an extension of a Cantor set.*

Recall that a subset U of a topological space is locally connected if every point of U admits a neighbourhood basis consisting of open connected subsets.

Theorem 1.4. *[8] The suspension v_f of f satisfies one of the following conditions:*
1) each orbit closure of v_f is toral.
2) there is a minimal set which is not locally connected.

We state the results for $\mathbb{R}$-actions. Let v be a continuous $\mathbb{R}$-action on a connected orientable closed surface S. Denote by LD the union of locally dense orbits.

Theorem 1.5. *[11] The* $\mathbb{R}$*-action* v *is non-wandering if and only if* $\overline{LD \sqcup \mathrm{Per}(v)} \cup \mathrm{Sing}(v) = S$*. In particular, if* v *is non-wandering, then* $\mathrm{Per}(v)$ *is open and there are no exceptional orbits.*

In [1] and [4], it is showed that the following properties are equivalent for an action of a finitely generated group G on either a compact zero-dimensional space or a graph X: (1) (G, X) is pointwise recurrent. (2) (G, X) is pointwise almost periodic. (3) (G, X) is R-closed. Now we state $\mathbb{R}$-actions on surfaces.

Theorem 1.6. *[6] The following are equivalent:*
1) v *is pointwise recurrent.*
2) v *is pointwise almost periodic.*
3) v *is either minimal or pointwise periodic.*

Theorem 1.7. *[6] Suppose* v *is neither identical nor minimal. Then* v *is* R*-closed if and only if* v *consists of periodic orbits and at most two centers.*

We state the results for foliations.

Theorem 1.8. *[6] Let* $\mathcal{F}$ *a continuous codimension one foliation on a closed connected manifold. The following are equivalent:*
1) $\mathcal{F}$ *is pointwise almost periodic.*
2) $\mathcal{F}$ *is* R*-closed.*
3) $\mathcal{F}$ *is minimal or compact.*

Note that, for a closed connected manifold M, the set of codimension two foliations on M which are minimal or compact is a proper subset of the set of R-closed codimension two foliations on M [9].

References

[1] Auslander, J.; Glasner, E.; Weiss, B., On recurrence in zero dimensional flows in *Forum Math.* **19** (2007), no. 1, 107–114.

[2] Jäger T., Kwakkel F., Passeggi A. A Classification of Minimal Sets of Torus Homeomorphisms in *Math. Z.* (2012), DOI 10.1007/s00209-012-1076-y.

[3] Koropecki, A., Aperiodic invariant continua for surface homeomorphisms in *Math. Z.* **266** (2010), no. 1, 229–236.

[4] Hattab, H., Pointwise recurrent one-dimensional flows in *Dyn. Syst.* **26** (2011), no. 1, 77–83.

[5] Mason, W. K., Weakly almost periodic homeomorphisms of the two sphere in *Pacific J. Math.* **48** (1973), 185–196.

[6] Yokoyama, T., Recurrence, pointwise almost periodicity and orbit closure relation for flows and foliations arXiv:1205.3635.

[7] Yokoyama, T., *R*-closed homeomorphisms on surfaces arXiv:1205.3634.

[8] Yokoyama, T., Minimal sets of *R*-closed surface homeomorphisms in *C. R. Acad. Sci. Paris, Ser. I 350 (2012) 1051–1053.*

[9] Yokoyama, T., R-closedness and Upper semicontinuity arXiv:1209.0166.

[10] Yokoyama, T., Recurrent and Non-wandering properties for decompositons arXiv:1210.7589.

[11] Yokoyama, T., Topological characterisations for non-wandering surface flows arXiv:1210.7623.

Department of Mathematics, Hokkaido University
Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan
E-mail: yokoyama@math.sci.hokudai.ac.jp