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Metric diffusion along compact foliations

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1. Wasserstein metric

The Wasserstein distance $d_{\mathcal{W}}$ of Borel probability measures μ and ν on Polish space X (complete separable metric space) endowed with a metric d is defined by

$$d_{\mathcal{W}}(\mu, \nu) = \inf \int_{M\times M} d(x, y) d\rho$$

where infimum is taken over all Borel probability measures ρ on $X \times X$ satisfying for any measurable sets $A, B \subset X$

$$\rho(A \times X) = \mu(A),$$
$$\rho(X \times B) = \nu(B).$$

A measure ρ is called *a coupling* of μ and ν. The set $\mathcal{P}(M)$ of all Borel probability measures with finite first moment endowed with $d_{\mathcal{W}}$ is a metric space. Moreover, $d_{\mathcal{W}}$ metrizes the weak-$*$ topology. The metric $d_{\mathcal{W}}$ comes from the Monge-Kantorovich optimal transportation problem [10] [11]. One can find that

Theorem 1.1. *[11] For any two Borel probability there exists a coupling ρ for which the Wasserstein distance is realized.*

One should notice that the Wasserstein distance $d_{\mathcal{W}}(\delta_x, \delta_y)$ of Dirac masses concentrated in points $x, y \in M$ is equal to the distance $d(x, y)$. This fact follows directly from the fact, that $\delta_{(x,y)}$ is the only coupling of δ_x and δ_y.

Let $\Delta^k = \{(t_1, \ldots, t_k) \in \mathbb{R}^k : t_j \geq 0, \sum_j t_j = 1\}$.

Proposition 1.2. *The set*

$$\mathcal{D}(M) = \{\mu \in \mathcal{P}(M) : \mu = \sum_{i=1}^{k} t_k \delta_{x_k},\ (t_1, \ldots, t_k) \in \Delta^k,\ x_1, \ldots, x_k \in M\}$$

is dense in $\mathcal{P}(M)$.

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2. Harmonic measures and heat diffusion

Let $(M, \mathcal{F}, g)$ be a smooth closed oriented foliated manifold equipped with a Riemannian metric g and Laplace-Beltrami operator Δ defined by

$$\Delta f = \mathrm{div} \nabla f.$$

Let $\Delta_{\mathcal{F}}$ be *foliated Laplace-Beltrami operator* [2] [13] given by

$$\Delta_{\mathcal{F}} f(x) = \Delta_{L_x} f(x), \quad x \in M,$$

where L_x is a leaf through x, and Δ_L is Laplace-Beltrami operator on $(L, g|L)$. The operator $\Delta_{\mathcal{F}}$ acts on bounded measurable functions, which are C^2-smooth along the leaves.

Let us recall that a probability measure μ on $(M, \mathcal{F}, g)$ is called *harmonic* if for any $f : M \to \mathbb{R}$

$$\int_M \Delta_{\mathcal{F}} f d\mu = 0.$$

Theorem 2.1. *[8] [1] On any compact foliated Riemannian manifold, harmonic probability measures exist.*

One can associate with the operator $\Delta_{\mathcal{F}}$ the one-parameter semigroup D_t, $t \geq 0$, of *heat diffusion operators* characterized by

$$d_0 = \text{ id}, \ D_{t+s} = D_t \circ D_s, \ \frac{d}{dt} D_t|_{t=0} = \Delta_{\mathcal{F}}.$$

D_t restricted to a leaf $L \in \mathcal{F}$ coincides with the heat diffusion operators on L, which are given by

$$D_t f(x) = \int_{L_x} f(y) p(x, y; t) d \ \mathrm{vol}_{L_x}, \tag{2.2}$$

where $p(\cdot, \cdot; t)$ is a *foliated heat kernel* [2] on $(M, \mathcal{F})$. The foliated heat kernel is nonnegative and for any $t > 0$ satisfies

$$\int_{L_x} p(x, y; t) d \ \mathrm{vol}_{L_x} = 1.$$

Let μ be a probability measure on M. According to [2, 13], one can define *the diffused measure* $D_t \mu$ by the formula

$$\int f d D_t \mu = \int D_t f d\mu,$$

where f is any bounded measurable function on M. A measure μ is called *diffusion invariant* when $D_t \mu = \mu$ for all $t > 0$.

3. Diffused metric

Let $(M, \mathcal{F}, g)$ be a smooth compact foliated manifold equipped with a Riemannian metric g and carrying foliation $\mathcal{F}$. Let δ_t denotes the Dirac measure at point x. For $t > 0$ the metric

$$D_t d(x, y) = d_{\mathcal{W}}(D_t \delta_x, D_t \delta_y) \tag{3.1}$$

will be called *the metric diffused along the foliation $\mathcal{F}$ at time t*. Since $d_{\mathcal{W}}(\delta_x, \delta_y) = d(x, y)$ for any $x, y \in M$ and $D_0 =$ id, we see that $D_0 d$ coincides the metric d. The metric space $(M, D_t d)$ will be denoted by M_t.

Theorem 3.2. *For any $s, t \geq 0$, metrics $D_t d$ and $D_s d$ are equivalent.*

4. Metric diffusion for compact foliations of dimension one

First, we recall some facts about compact foliations, i.e. foliations with all leaves compact. The topology of the leaf space of a compact foliation $\mathcal{F}$ on a compact manifold M does not have to be Hausdorff. Examples of such foliations were presented by Epstein and Vogt [7], Sullivan [9] and Vogt [12].

The following result describes the topology of a compact foliation in few equivalent conditions. First, denote by $\pi : M \to \mathcal{L}$ the quotient projection defined by $\pi(x) = L_x$, where $\mathcal{L}$ denotes the space of leaves of a foliation $\mathcal{F}$, i.e., a quotient space of the equivalence relation $x \sim y$ if and only if $L_x = L_y$, where L_z denotes the leaf through z.

Theorem 4.1. *[6] The following conditions are equivalent:*

1. *π is a closed map.*
2. *π maps compact sets onto closed sets.*
3. *Each leaf has arbitrarily small saturated neighborhoods.*
4. *$\mathcal{L}$ with quotient topology is Hausdorff.*
5. *If $K \subseteq M$ is compact, then the saturation of K is also compact.*

Let $G_{\mathcal{F}}$ be the set of all points $x \in M$ near which the volume function is bounded, i.e., $x \in G_{\mathcal{F}}$ if and only if there exists an open neighborhood U of x such that the volumes of all leaves passing through U are uniformly bounded. The set $G_{\mathcal{F}}$ is called *the good set* of the foliation $\mathcal{F}$. Due to [5], $G_{\mathcal{F}}$ is open, saturated, and dense in M and all holonomy groups of leaves contained in $G_{\mathcal{F}}$ are finite. The complement $B_{\mathcal{F}} = M \setminus G_{\mathcal{F}}$ of the good set is called *the bad set*. It follows directly from the definition of the good set and Theorem 4.1 that foliations with empty bad set have a volume of leaves commonly bounded.

One of the most important results about compact foliations is the following:

Theorem 4.2. *[4] Let us suppose that M is a smooth compact Riemannian manifold which is foliated by compact foliation of co-dimension one or two. There is an upper bound of the volumes of the leaves of M.*

Let $\mathcal{F}$ be a compact foliation on a compact Riemannian manifold (M, g) with the volume of leaves commonly bounded above. The classical result says that on a compact manifold M the heat is evenly distributed over M while time is tending to infinity. More precisely,

Theorem 4.3. *[3] For any $f \in L^2(M)$, the function $D_t f$ converges uniformly, as t goes to the infinity, to a harmonic function on M. Since M is compact, the limit function is a constant.*

Let $L, L' \in \mathcal{F}$ be two leaves. One can define the metric $\rho_{\,\mathrm{vol}}$ in the space of leaves by

$$\rho_{\,\mathrm{vol}}(L, L') = d_{\mathcal{W}}(\overline{\,\mathrm{vol}}(L), \overline{\,\mathrm{vol}}(L')),$$

where $\overline{\,\mathrm{vol}}(F)$ denotes the normalized volume of the leaf F.

We will now restrict to the compact foliations of dimension 1. We will study the convergence in the Wasserstein-Hausdorff topology of the natural isometric embeddings $\iota : M_t \to \mathcal{P}(M)$ defined by

$$\iota_t(x) = D_t \delta_x.$$

Precisely speeking, $\iota_t(M, D_t d)$ is a compact subset of $\mathcal{P}(M)$, while we define *the Wasserstein-Hausdorff distance of diffused metrics* by

$$d_{\mathcal{WH}}(M_t, M_s) = (d_{\mathcal{W}})_H(\iota_t(M), \iota_s(M)),$$

where $(d_{\mathcal{W}})_H$ denotes the Hausdorff distance of closed subsets of $\mathcal{P}(M)$.

Theorem 4.4. *The Gromov-Hausdorff limit of a diffused foliation with empty bad set is isometric to the space of leaves equipped with the metric $\rho_{\,vol}$.*

The following example visualizes that in the above Theorem the assumption on the bad set is necessary.

EXAMPLE 4.5. Following [12], let G be a topological group, while $\gamma : [0, 2\pi] \to G$ a closed curve. One can define a one dimensional foliation $\mathcal{F}(\gamma)$ on $S^1 \times G$ filling it by closed curves as follows:

Through a point $(t,x) \in S^1 \times G$ passes a curve

$$[0,2\pi] \ni s \mapsto (s, \gamma(s)\gamma(t)^{-1}x).$$

Leaves of $\mathcal{F}(\gamma)$ are the fibers of a trivial bundle over G with a fiber S^1. Moreover, if G is a Lie group then $\mathcal{F}(\gamma)$ is a C^r-foliation if only γ is a C^r-curve.

Consider as a Lie group a sphere $S^3 = \{(z,w) \in \mathbb{C}^2 : z\bar{z} + w\bar{w} = 1\}$ with multiplication defined by

$$(a,b) \cdot (c,d) = (ac - b\bar{d}, ad + b\bar{c}).$$

The first step is to define, for any $\tau \in (0,1]$, a curve $\gamma_\tau : [0,2\pi] \to S^3$ as follows:

1. if $\tau = \frac{1}{2n+1} - t$, $0 \le t \le \frac{1}{(2n+1)(2n+2)} = a_n$, $n = 0,1,2,\dots$ then
$$\gamma_\tau(s) = (\sqrt{1 - (\frac{t}{a_n})^2}\, e^{ins}, \frac{t}{a_n}\, e^{ins}), \quad s \in [0,2\pi];$$
2. if $\tau = \frac{1}{2n} - t$, $0 \le t \le \frac{1}{2n(2n+1)} = b_n$, $n = 1,2,\dots$ then
$$\gamma_\tau(s) = (\frac{t}{b_n}\, e^{ins}, \sqrt{1 - (\frac{t}{b_n})^2}\, e^{i(n+1)s}), \quad s \in [0,2\pi].$$

One can easily check that the family γ_τ is continuous.

Next step is to foliate $(0,1] \times S^1 \times S^3$ foliating, for given $\tau \in (0,1]$, the set $\{\tau\} \times S^1 \times S^3$ by $\mathcal{F}(\gamma_\tau)$. Directly from the definition of $\mathcal{F}(\gamma_\tau)$, one can see that the length of leaves tends to infinity, and the length of the S^1 component of the vector tangent to a leaf goes to 0 while $\tau \to 0$. Moreover, γ_τ converge tangentially to the left co-sets of closed 1-parameter subgroup

$$H = \{(e^{is}, 0), s \in [0,2\pi]\}.$$

Complementing the foliation of $M = [0,1] \times S^1 \times S^3$ by a foliation of $\{0\} \times S^1 \times S^3$ by leaves of the form

$$\{0\} \times \{t\} \times H \cdot g, \quad g \in S^3, t \in S^1$$

we obtain 1-dimensional foliation $\tilde{\mathcal{F}}$ of $[0,1] \times S^1 \times S^3$ with nonempty bad set.

Now, we introduce a modification of $\tilde{\mathcal{F}}$ to obtain our target foliation.

Let $h : [0,2\pi] \to [0,2\pi]$ be a increasing function with the graph as on the Figure 1

Next, let $\bar{h} : [0,1] \times [0,2\pi] \to [0,2\pi]$ be a smooth homotopy from identity to h, that is $\bar{h}(t,s) = (1-t)s + th(s)$. Define a modificating function $\tilde{h} : [0,1] \times [0,2\pi] \to [0,2\pi]$ by the formula

$$\tilde{h}(t,s) = \begin{cases} \bar{h}(2t,s) & \text{for } t \in [0,\frac{1}{2}], \\ \bar{h}(-2t+2,s) & \text{for } t \in [\frac{1}{2},1]. \end{cases}$$

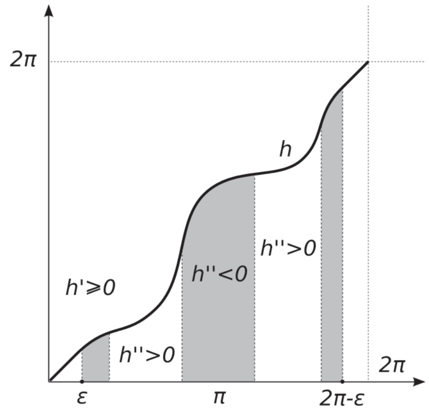

Figure 1: A modificating function.

Having $\tilde{h}$, we define mappings $H_n : [0,1] \times S^1 \times S^3 \to [0,1] \times S^1 \times S^3$ by

$$\tilde{H}_n(\tau, s, x) = \begin{cases} (\tau, \tilde{h}(n(n+1)\tau - n), s), x) \\ \qquad\qquad \text{for } (\tau, s, x) \in [\frac{1}{2n+2}, \frac{1}{2n+1}] \times S^1 \times S^3, \\ (\tau, s, x) \qquad \text{otherwise.} \end{cases}$$

Note that H_n changes $\tilde{\mathcal{F}}$ only on the set

$$(\tau, s, x) \in [\frac{1}{2n+2}, \frac{1}{2n+1}] \times S^1 \times S^3$$

and leaves it unchanged everywhere else.

Let us modify the foliation $\tilde{\mathcal{F}}$ as follows:

For $n_1 = 1$ set $\mathcal{F}_1 = (H_1)_* \tilde{\mathcal{F}}$. Next, choose $\theta_1 > 0$ such that for all $\theta > \theta_1$ and all $p = (\tau, s, x) \in [\frac{1}{2n_1+2}, 1] \times S^1 \times S^3$

$$d_{\mathcal{W}}(D_{\theta_1}\delta_p, \overline{\text{vol}}(L_p)) < \frac{1}{2^{n_1}}.$$

Suppose that we have choosen $n_k > n_{k-1}$ and $\theta_k > \theta_{k-1}$ such that for foliation

$$\mathcal{F}_k = (H_k \circ \cdots \circ H_1)_* \tilde{\mathcal{F}}$$

and all $p = (\tau, s, x) \in [\frac{1}{2(n_k+1)}, 1] \times S^1 \times S^3$

$$d_{\mathcal{W}}(D_{\theta_k}\delta_p, \overline{\text{vol}}(L_p)) < \frac{1}{2^{n_k}}.$$

Let us choose $n_{k+1} > n_k$ for which all leaves of $\mathcal{F}_{k+1} = (H_{k+1})_* \mathcal{F}_k$ passing through $p = (\tau, s, x) \in [0, \frac{1}{n_{k+1}}] \times S^1 \times S^3$ satisfy

$$d_{\mathcal{W}}(D_{\theta_k}\delta_p, \overline{\text{vol}}(L_{(0,s,x)})) < \frac{1}{2^k}.$$

Finally foliation $\mathcal{F}$ as $(\cdots H_n \circ \cdots \circ H_1)_* \tilde{\mathcal{F}}$ and consider the Riemannian metric d induced from $\mathbb{R}^7$ equipped wih $\mathcal{F}$ on M.

Theorem 4.6. *The family of $(M, \mathcal{F}, D_t d)$ does not satisfies the Cauchy condition in Wasserstein-Hausdorff topology. Namely, there exists $\epsilon_0 > 0$ such that for any $T > 0$ one can find $\theta, \lambda > T$ satisfying*

$$d_{\mathcal{WH}}(M_\theta, M_\lambda) > \epsilon_0.$$

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