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Prevalence of non-uniform hyperbolicity at the first bifurcation of Hénon-like families

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Hyperbolicity and structural stability are key concepts in the development of the theory of dynamical systems. Nowadays, it is known that these two concepts are essentially equivalent to each other, at least for C^1 diffeomorphisms or flows of a compact manifold. Then, a fundamental problem in the bifurcation theory is to study transitions from hyperbolic to non hyperbolic regimes. When the loss of hyperbolicity is due to the formation of a cycle (i.e., a configuration in the phase space involving non-transverse intersections between invariant manifolds), an incredibly rich array of complicated behaviors is unleashed by the unfolding of the cycle (See e.g. [12] and the references therein). Many important aspects of this complexity are poorly understood.

To study bifurcations of diffeomorphisms, we work within a framework set up by Palis: consider arcs of diffeomorphisms losing their hyperbolicity through generic bifurcations, and analyze which dynamical phenomena are more frequently displayed (in the sense of the Lebesgue measure in parameter space) in the sequel of the bifurcation. More precisely, let $\{\varphi_a\}_{a\in\mathbb{R}}$ be a parametrized family of diffeomorphisms which undergoes a first bifurcation at $a = a^*$, i.e., φ_a is hyperbolic for $a > a^*$, and φ_{a^*} has a cycle. We assume $\{\varphi_a\}_{a\in\mathbb{R}}$ unfolds the cycle generically. A dynamical phenomenon $\mathcal{P}$ is *prevalent* at a^* if

$$\liminf_{n\to\infty} \frac{1}{\varepsilon}\mathrm{Leb}\{a \in [a^* - \varepsilon, a^*] \colon \varphi_a \text{ displays } \mathcal{P}\} > 0,$$

where Leb denotes the one-dimensional Lebesgue measure.

Particularly important is the prevalence of hyperbolicity. The pioneering work in this direction is due to Newhouse and Palis [8], on the bifurcation of Morse-Smale diffeomorphisms. The prevalence of hyperbolicity in arcs of surface diffeomorphisms which are not Morse-Smale has been studied in the literature [7, 10, 11, 13, 14]. However, even with all these and other subsequent developments, including [15, 16], we still lack a good understanding as to in which case the hyperbolicity becomes prevalent.

In [7, 10, 11, 13, 14], unfoldings of tangencies of surface diffeomorphisms associated to basic sets have been treated. One key aspect of these models is that the orbit of tangency at the first bifurcation is not contained in the limit set. This implies a global control on new orbits added to the underlying basic set, and moreover allows one to use its invariant foliations to translate dynamical problems to the problem on how two Cantor sets intersect each other. Then, the prevalence of hyperbolicity is related to the Hausdorff dimension of the limit set. This argument is not viable, if the orbit of tangency, responsible for the loss of the stability of the system, is contained in the limit set. Let us call such a first bifurcation an *internal tangency bifurcation.*

We are concerned with an arc $\{f_a\}_{a\in\mathbb{R}}$ of planar diffeomorphisms of the form

$$f_a(x,y) = (1-ax^2, 0) + b\cdot\Phi(a,b,x,y), \quad 0 < b \ll 1.$$

Here Φ is bounded continuous in (a,b,x,y) and C^4 in (a,x,y). This particular arc, often called an "Hénon-like family", is embedded in generic one-parameter unfoldings of quadratic homoclinic tangencies associated to dissipative saddles [6], and so is relevant in the investigation of structurally unstable surface diffeomorphisms.

Let Ω_a denote the non wandering set of f_a, which is a compact f_a-invariant set. It is known [5] that for sufficiently large $a>0$, f_a is Smale's horseshoe map and Ω_a admits a hyperbolic splitting into uniformly contracting and expanding subspaces. As a decreases, the infimum of the angles between these two subspaces gets smaller, and the hyperbolic splitting disappears at a certain parameter. This first bifurcation is an internal tangency bifurcation. Namely, for sufficiently small $b>0$ there exists a parameter $a^*=a^*(b)$ near 2 with the following properties [1, 2, 3, 5].:

- if $a>a^*$, then Ω_a is a hyperbolic set, i.e., there exist constants $C>0$, $\xi\in(0,1)$ and at each $x\in\Omega_a$ a non-trivial decomposition $T_x\mathbb{R}^2 = E^s_x\oplus E^u_x$ with the invariance property such that $\|D_xf^n_a|E^s_x\|\leq C\xi^n$ and $\|D_xf^{-n}_a|E^u_x\|\leq C\xi^n$ for every $n\geq 0$;
- there is a quadratic tangency between stable and unstable manifolds of the fixed points of f_{a^*}. The orbit of this tangency at $a=a^*$ is accumulated by transverse homoclinic points, and thus it is contained in the limit set.

The orbit of tangency of f_{a^*} is in fact unique, and $\{f_a\}_{a\in\mathbb{R}}$ unfolds this tangency generically. The next theorem gives a partial description of prevalent dynamics at $a=a^*$.

Theorem 1. *For sufficiently small $b>0$ there exist $\varepsilon_0=\varepsilon_0(b)>0$ and a set $\Delta\subset[a^*-\varepsilon_0,a^*]$ of a-values containing a^* with the following properties:*

(a) $\lim_{\varepsilon\to+0}(1/\varepsilon)Leb(\Delta\cap[a^*-\varepsilon,a^*])=1$;

(b) if $a\in\Delta$, then the Lebesgue measure of the set

$$K^+_a := \{x\in\mathbb{R}^2\colon \{f^n_ax\}_{n\in\mathbb{N}} \text{ is bounded}\}$$

is zero. In particular, for Lebesgue almost every $x \in \mathbb{R}^2$, $|f_a^n x| \to \infty$ as $n \to \infty$.

In addition, if $a \in \Delta$ then f_a is transitive on Ω_a. In other words, for "most" diffeomorphisms immediately right after the first bifurcation, the topological dynamics is similar to that of Smale's horseshoe before the bifurcation.

We suspect that the dynamics is non hyperbolic for all, or "most" parameters in Δ. Nevertheless, the proof of the above theorem tells us that the dynamics of f_a, $a \in \Delta$ is fairly structured, and this may yield a weak form of hyperbolicity. A natural question then is the following:

To what extent the dynamics is hyperbolic for $a \in \Delta$?

The main result of this paper gives one answer for this question. For measuring the extent of hyperbolicity we estimate Lyapunov exponents, the asymptotic exponential rates at which nearby orbits are separated (or draw together).

Let us say that a point $x \in \Omega_a$ is *regular* if there exist number(s) $\chi_1 < \cdots < \chi_{r(x)}$ and a decomposition $T_x\mathbb{R}^2 = E_1(x) \oplus \cdots \oplus E_{r(x)}(x)$ such that for every $v \in E_i(x) \setminus \{0\}$,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f_a^n v\| = \chi_i(x) \quad \text{and}$$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\det D_x f_a^n| = \sum_{i=1}^{r(x)} \chi_i(x) \mathrm{dim} E_i(x).$$

By the theorem of Oseledec [9], the set of regular points has total probability. If μ is ergodic, then the functions $x \mapsto r(x)$, $\lambda_i(x)$ and $\mathrm{dim}E_i(x)$ are invariant along orbits, and so are constant μ-a.e. From this and the Ergodic Theorem, one of the following holds for each ergodic μ:

- there exist two numbers $\chi^s(\mu) < \chi^u(\mu)$, and for μ-a.e. $x \in \Omega_a$ a decomposition $T_x\mathbb{R}^2 = E_x^s \oplus E_x^u$ such that for any $v^\sigma \in E_x^\sigma \setminus \{0\}$ and $\sigma = s,\, u$,
$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f_a^n v\| = \chi^\sigma(\mu) \quad \text{and}$$
$$\int \log |\det Df_a| d\mu = \chi^s(\mu) + \chi^u(\mu);$$
- there exists $\chi(\mu) \in \mathbb{R}$ such that for μ-a.e. $x \in \Omega_a$ and all $v \in T_x\mathbb{R}^2 \setminus \{0\}$,
$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f_a^n v\| = \chi(\mu) \quad \text{and}$$

$$\int \log|\det Df_a| d\mu = 2\chi(\mu).$$

The number(s) $\chi^s(\mu)$ and $\chi^u(\mu)$, or $\chi(\mu)$ is called a *Lyapunov exponent(s)* of μ.

Let $\mathcal{M}^e(f_a)$ denote the set of f_a-invariant Borel probability measures which are ergodic. We call $\mu \in \mathcal{M}^e(f_a)$ a *hyperbolic measure* if μ has two Lyapunov exponents $\chi^s(\mu)$, $\chi^u(\mu)$ with $\chi^s(\mu) < 0 < \chi^u(\mu)$. Our main theorem indicates a strong form of non-uniform hyperbolicty for $a \in \Delta$.

Theorem 2. *For sufficiently small $b > 0$, the following holds for all $a \in \Delta$:*

(a) any $\mu \in \mathcal{M}^e(f_a)$ is a hyperbolic measure;

(b) for each $\mu \in \mathcal{M}^e(f_a)$,

$$\chi^s(\mu) < \frac{1}{3}\log b < 0 < \frac{1}{4}\log 2 < \chi^u(\mu).$$

It must be emphasized that this kind of uniform bounds on Lyapunov exponents of ergodic measures are compatible with the non hyperbolicity of the system, and therefore, Theorem A does not imply the uniform hyperbolicity for $a \in \Delta$. Indeed, $a^* \in \Delta$ and f_{a^*} is genuinely non hyperbolic, due to the existence of tangencies. See [3, 4] for the first examples of non hyperbolic surface diffeomorphisms of this kind. As already mentioned, we suspect that the dynamics is non hyperbolic for all, or "most" parameters in Δ.

Little is known on the prevalence of hyperbolicity at internal tangency bifurcations. The only previously known result in this direction is due to Rios [15], on certain horseshoes in the plane with three branches. However, certain hypotheses in [15] on expansion/contraction rates and curvatures of invariant manifolds near the tangency, are no longer true for $\{f_a\}_{a\in\mathbb{R}}$ due to the strong dissipation.

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