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The mixed scalar curvature flow and harmonic foliations

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A flow of metrics, g_t, on a manifold is a solution of evolution equation $\partial_t g = S(g)$, where $S(g)$ is a symmetric $(0, 2)$-tensor usually related to some kind of curvature. The mixed sectional curvature of a foliated manifold $(M, \mathcal{F})$ regulates the deviation of leaves along the leaf geodesics. (In the language of mechanics it measures the rate of relative acceleration of two particles moving forward on neighboring geodesics). Let $\{\varepsilon_\alpha, e_i\}_{\alpha\le p, i\le n}$ be a local orthonormal frame on TM adapted to $T\mathcal{F}$ and the orthogonal distribution $\mathcal{D} := T\mathcal{F}^\perp$.

The *mixed scalar curvature* is defined by $\mathrm{Sc}_{\rm mix} = \sum_{i=1}^n \sum_{\alpha=1}^p R(\varepsilon_\alpha, e_i, \varepsilon_\alpha, e_i)$, where R is the Riemannain curvature. For a codimension-one foliation with a unit normal N, we have $\mathrm{Sc}_{\rm mix} = \mathrm{Ric}(N, N)$. For a surface (M^2, g), i.e., $n = p = 1$, we obtain $\mathrm{Sc}_{\rm mix} = K$ – the gaussian curvature.

We study the flow of metrics on a foliation, whose velocity along $\mathcal{D}$ is proportional to $\mathrm{Sc}_{\rm mix}$:

$$\partial_t g = -2(\mathrm{Sc}_{\rm mix}(g) - \Phi)\hat{g}. \tag{1}$$

Here $\Phi\colon M \to \mathbb{R}$ is leaf-wise constant. The *$\mathcal{D}$-truncated metric tensor* $\hat{g}$ is given by $\hat{g}(X_1, X_2) = g(X_1, X_2)$ and $\hat{g}(Y, \cdot\,) = 0$ for $X_i \in \mathcal{D}$, $Y \in T\mathcal{F}$. We show relations of (1) with Burgers equation (the prototype for non-linear advection-diffusion processes in gas and fluid dynamics) and Schrödinger heat equation (which is central to all of quantum mechanics).

Let $h_\mathcal{F}$, h be the second fundamental forms and $H_\mathcal{F}$, H the mean curvature vectors of $T\mathcal{F}$ and the distribution $\mathcal{D}$, respectively. Also denote T the integrability tensor of $\mathcal{D}$. Then, see [2],

$$\mathrm{Sc}_{\rm mix}(g) = \mathrm{div}(H + H_\mathcal{F}) + \|H\|^2 + \|T\|^2 - \|h\|^2 + \|H_\mathcal{F}\|^2 - \|h_\mathcal{F}\|^2. \tag{2}$$

The flow (1) preserves total geodesy (i.e. $h_\mathcal{F} = 0$) and *harmonicity* (i.e. $H_\mathcal{F} = 0$) of foliations and is used to examine the question [1]: *Which foliations admit a metric with a given property of* $\mathrm{Sc}_{\rm mix}$ (*e.g., positive or negative*)? Suppose that the leaves of $\mathcal{F}$ are compact minimal submanifolds. We observe that (1) yields the leaf-wise evolution equation for the vector field H:

$$\partial_t H + \nabla^\mathcal{F} g(H, H) = n\nabla^\mathcal{F}(\mathrm{Div}_\mathcal{F} H) + n\nabla^\mathcal{F}(\|T\|_g^2 - \|h_\mathcal{F}\|_g^2 - n\beta_\mathcal{D}). \tag{3}$$

The function $\beta_{\mathcal{D}} := n^{-2}\big(n\|h\|^2 - \|H\|^2\big) \ge 0$ is time-independent, it serves as a measure of "*non-umbilicity*" of $\mathcal{D}$, since $\beta_{\mathcal{D}} = 0$ for totally umbilical $\mathcal{D}$. For $\dim\mathcal{F} = 1$ we have $\beta_{\mathcal{D}} = n^{-2}\sum_{i<j}(k_i - k_j)^2$, where k_i are the principal curvatures of $\mathcal{D}$.

Suppose that $H_0 = -n\nabla^{\mathcal{F}}(\log u_0)$ (leaf-wise conservative) for a function $u_0 > 0$.
If $\|T\|_{g_0} > \|h_{\mathcal{F}}\|_{g_0}$ then its potential obeys the leaf-wise non-linear Schrödinger heat equation

$$(1/n)\partial_t u = \Delta_{\mathcal{F}} u + (\beta_{\mathcal{D}} + \Phi/n)u - (\Psi/n)u^{-3}, \qquad u(\,\cdot\,,0) = u_0, \tag{4}$$

where $\Psi := u_0^4(\|T\|^2_{g_0} - \|h_{\mathcal{F}}\|^2_{g_0})$, moreover, the solution obeys $u = \Psi^{1/4}(\|T\|^2_{g_t} - \|h_{\mathcal{F}}\|^2_{g_t})^{-1/4}$.

If $\Psi \equiv 0$ (e.g., $T(g_0) = 0$ and $h_{\mathcal{F}}(g_0) = 0$) then (3) reduces to a forced *Burgers equation*

$$\partial_t H + \nabla^{\mathcal{F}} g(H,H) = n\nabla^{\mathcal{F}}(\mathrm{Div}_{\mathcal{F}} H) - n^2\nabla^{\mathcal{F}}\beta_{\mathcal{D}}, \tag{5}$$

moreover, the leaf-wise potential function for H may be chosen as a solution of the linear PDE $(1/n)\partial_t u = \Delta_{\mathcal{F}} u + \beta_{\mathcal{D}} u$, $u(\cdot,0) = u_0$. The first eigenvalue $\lambda_0 \le 0$ of *Schrödinger operator* $\mathcal{H}(u) = -\Delta_{\mathcal{F}} u - \beta_{\mathcal{D}} u$ corresponds to the unit L_2-norm eigenfunction $e_0 > 0$ (called the *ground state*). Under certain conditions (on any leaf F)
(6)

$$\Phi > -n\beta_{\mathcal{D}}, \quad |n\lambda_0 + \Phi| \ge \max_F(\|T\|^2_{g_0} - \|h_{\mathcal{F}}\|^2_{g_0})\big(\max_F(u_0/e_0)/\min_F(u_0/e_0)\big)^4$$

the asymptotic behavior of solutions to (4) is the same as for the linear equation, when (5) has a single-point global attractor: $H_t \to -n\nabla^{\mathcal{F}}(\log e_0)$ as $t \to \infty$. Using the scalar maximum principle, we show that there exists a positive solution $\tilde u$ of the linear PDE $(1/n)\partial_t\tilde u = \Delta_{\mathcal{F}}\tilde u + (\beta_{\mathcal{D}} + \lambda_0)\tilde u$ such that for any $\alpha \in \big(0, \min\{\lambda_1 - \lambda_0, 4|\lambda_0|\}\big)$ and $k \in \mathbb{N}$ the following hold:

(i) $u = e^{-\lambda_0 t}(\tilde u + \theta(x,t))$, where $\|\theta(\,\cdot\,,t)\|_{C^k} = O(e^{-\alpha t})$ as $t \to \infty$;
(ii) $\nabla^{\mathcal{F}}(\log u) = \nabla^{\mathcal{F}}(\log e_0) + \theta_1(x,t)$, where $\|\theta_1(\,\cdot\,,t)\|_{C^k} = O(e^{-\alpha t})$ as $t \to \infty$.

In this case, (1) has a unique global solution g_t ($t \ge 0$), whose $\mathrm{Sc}_{\mathrm{mix}}$ converges exponentially to $n\lambda_0 \le 0$. The metrics are smooth on M when all leaves are compact and have finite holonomy group. After rescaling of metrics on $\mathcal{D}$, we also obtain convergence to a metric with $\mathrm{Sc}_{\mathrm{mix}} > 0$.

Proposition 1. *Let (M,g) be endowed with a harmonic compact foliation $\mathcal{F}$. Suppose that $\|h_{\mathcal{F}}\|_g < \|T\|_g$ and $H = -n\nabla^{\mathcal{F}}(\log u_0)$ for a function $u_0 > 0$.*

(i) *If $\lambda_0 < 0$ then there exists $\mathcal{D}$-conformal to g metric $\bar g$ with* $\mathrm{Sc}_{\mathrm{mix}}(\bar g) < 0$.

(ii) *If* $\lambda_0 > -\frac{1}{n}(\frac{u_0}{\tilde{u}_0 e_0})^4(\|T\|_g^2 - \|h_{\mathcal{F}}\|_g^2)$ *then there is* $\mathcal{D}$*-conformal to* g *metric* $\bar{g}$ *with* $\mathrm{Sc}_{\mathrm{mix}}(\bar{g}) > 0$.

For surfaces of revolution M_t: $[\rho(x,t)\cos\theta, \rho(x,t)\sin\theta, h(x)]$ $(0 \le x \le l$, $|\theta| \le \pi)$ with $(\rho_{,x})^2 + (h_{,x})^2 = 1$, (1) reads as $\partial_t g = -2(K(g) - \Phi)\hat{g}$. This yields the PDE $\partial_t \rho = \rho_{,xx} + \Phi\rho$. For $\Phi = \mathrm{const}$ and appropriate initial and end conditions for ρ, we have the following. If $\Phi < (\pi/l)^2$ then M_t converge to a surface with $K = \Phi$, and if $\Phi = (\pi/l)^2$ then $\lim_{t\to\infty} \rho(x,t) = A\sin(\pi x/l)$, and M_t converge to a surface with $K = \Phi$ (a sphere of radius l/π when $A = l/\pi$).

References

[1] V. Rovenski and L. Zelenko: *The mixed scalar curvature flow and harmonic foliations*, ArXiv:1303.0548, preprint, 20 pp. 2013.

[2] P. Walczak: *An integral formula for a Riemannian manifold with two orthogonal complementary distributions.* Colloq. Math. 58 (1990), 243–252.

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