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Almost Contact Confoliations and their Dimensionality Reductions

Atsuhide MORI

1. Introduction

The following two recent results suggest that the topology of codimension one foliations of high dimensional manifolds has two opposite possibilities. Namely, the absence/presence of leafwise symplectic structures could make it dissimilar/similar to the 3-dimensional topology of foliations.

Theorem 1.1 (Meigniez [3]). *Let $\mathcal{F}$ be a transversely oriented codimension one foliation of a closed $(n+3)$-manifold M^{n+3} which is just smooth. Then we can deform $\mathcal{F}$ to a minimal (all leaves dense) foliation $\mathcal{F}'$ such that $T\mathcal{F}'$ is homotopic to $T\mathcal{F}$ as a tangent hyperplane field on M^{n+3} $(n>0)$.*

Theorem 1.2 (Martínez Torres [2]). *Let $\mathcal{F}$ be an oriented codimension one foliation of an oriented closed $(2n+3)$-manifold M^{2n+3}. Suppose that there exists a closed 2-form ω on M with $\omega^{n+1}|T\mathcal{F}>0$. Then Donaldson-Auroux approximately holomorphic geometry provides a codimension $2n$ submanifold N^3 such that $\mathcal{G}=\mathcal{F}|N^3$ is a taut foliation and N^3 meets each leaf of $\mathcal{F}$ at a single leaf of $\mathcal{G}$ (i.e., $M^{2n+3}/\mathcal{F}=N^3/\mathcal{G}$).*

Mitsumatsu found another kind of leafwise symplectic foliation which has the same leaf space as a non-taut foliation of a 3-manifold.

Theorem 1.3 (Mitsumatsu [4]). *The Lawson foliation of S^5, which is a leafwise fattening of the Reeb foliation of S^3, admits a leafwise symplectic structure. (It is the restriction of a non-closed 2-form on S^5).*

The Eliashberg-Thurston 3-dimensional confoliation theory discretized the vast whole of foliations into contact structures. In [6], the author defined higher dimensional confoliations by means of almost contact geometry:

Definition 1.4. Let $([\alpha],[\omega])$ be the pair of conformal classes of a 1-form α and a 2-form ω on a closed oriented $(2n+1)$-manifold M^{2n+1}.

1. We say that $([\alpha],[\omega])$ (or $[\alpha]$ itself) is an almost contact structure if

it satisfies $[\alpha] \wedge [\omega]^n > 0$ (for some $[\omega]$).

2. We say that an almost contact structure is a contact structure (resp. a foliation) if $\ker \alpha$ is contact (resp. tangent to a foliation).
3. We say that an almost contact structure is a confoliation if it satisfies $[\alpha \wedge d\alpha^n] \geq 0$. If moreover it belongs to the closure of the space of contact structures or to the space of foliations (in the space of pairs $([\alpha], [\omega])$ with smooth topology), it is called a strict confoliation.

Though we omit the precise construction in this abstract, we fix a situation where we can obtain a family of higher dimensional strict confoliations which goes to a leafwise symplectic foliation (§2).

Outside the Donaldson-Auroux approximately holomorphic geometry, convex hypersurface theory due to Giroux is the most powerful tool in contact topology. In 3-manifold case, it can be considered as a contact version of sutured manifold theory due to Gabai (Honda-Kazez-Matić [1]). Further Honda's category theory regards a contact structure between convex surfaces as a morphism. This author is now trying to generalize this theory in order to understand the (perhaps proper and natural) affinity between high dimensional contact topology and 3-dimensional one (§3).

2. Confoliations

We start with the Thurston-Winkelnkemper-Giroux construction of contact structure on a closed $(2n+1)$-manifold M^{2n+1} equipped with a pagewise exact symplectic open-book structure $\mathcal{O}$. One may say that this construction is an extension (or a fattening) of a contact structure $\ker \mu$ on the binding $\mathcal{B}$ of $\mathcal{O}$ under the presence of exact symplectic filling pages. In [5], the author pointed out that we can further construct a family of contact structures convergent to a foliation $\mathcal{F}$ in the case where the Reeb field of μ is tangent to a Riemannian foliation $\mathcal{G}$ of $\mathcal{B}$ defined by a closed 1-form ν $(n > 1)$. The foliation $\mathcal{F}$ consists of a closed leaf $L = \mathcal{B} \times S^1$, page leaves coiling into L, and a trivial extension of $\mathcal{G}$ also coiling into L. (One might remember the Calabi conjecture theorem of Friedl-Vidussi.)

Theorem 2.1 ([6]). *Assume that the Reeb field of μ is tangent to $\ker \nu$. Suppose moreover that there exists a closed 2-form Ω on $\mathcal{B}$ such that*

1. *$\nu \wedge (d\mu + \varepsilon\Omega)^n > 0$ holds for small $\varepsilon > 0$, and*
2. *Ω extends to a closed 2-form on the page.*

Then we can construct a family of pairs (α_t, ω_t) on M^{2n+1} such that

1. *$\alpha_t \wedge (d\alpha_t)^n > 0$ for $0 \leq t < 1$,*
2. *$\omega_0 = d\alpha_0$, $\ker \alpha_1 = T\mathcal{F}$,*

3. $\alpha_t \wedge \omega_t^n > 0$ *for* $0 \leq t \leq 1$ *, and*
4. $\omega_1 | \mathcal{F}$ *is leafwise symplectic.*

Example 2.2. The Milnor fibration of $x^3 + y^3 + z^3 = 0$ on $\mathbb{C}^3$ naturally defines a pagewise exact symplectic open-book on (small) S^5. In this case Mitsumatsu [4] found the above 2-form Ω. Then the strict confoliations $([\alpha_t], [\omega_t])$ starts with the standard contact structure on S^5 and goes to the Lawson foliation with leafwise symplectic structure.

3. Convex hypersurfaces

Let Σ be a compact oriented hypersurface in a contact manifold. Suppose that, for a suitable representative $\alpha \in [\alpha]$, the sign of $d\alpha^n | T\Sigma$ defines a dividing $\Sigma \setminus \Gamma = \Sigma_+ \cup (-\Sigma_-)$ along a submanifold $\Gamma \subset \Sigma$ into strongly pseudo-convex domains $\Sigma_\pm$. Then Σ is called a convex hypersurface. (One may generalize this notion in various ways, e.g., for leafwise symplectic foliations, one may consider a union of pseudo-convex domains on leaves connected with cylindrical "sutures" transverse to the foliation.)

In [5], the author generalized the Lutz modification of 3-dimensional contact structure by using convex hypersurface theory. In general, this modification changes the contact structure drastically (indeed makes it non-fillable) and produces a convex hypersurface which contains a strongly pseudo-convex domain with disconnected boundary. We call such a domain on a convex hypersurface a Calabi hypersurface.

Question 3.1. Is there any Calabi hypersurface in $S^{2n+3} \subset \mathbb{C}^{n+2}$?

We consider a certain generalization of the convex version of Thurston-Bennequin inequality in higher dimension since the natural generalization of the usual inequality does not hold even locally. On the other hand, if a convex hypersurface violates the inequality, it contains a Calabi hypersurface. Here we notice that, while a surfaces in contact 3-manifolds are smoothly approximated by convex ones, that is not the case with hypersurfaces in higher dimension. Anyway the generalized Lutz modification produces a convex hypersurface which violates the inequality.

The existence problem of convex hypersurfaces is also interesting. First we see that the boundary of the standard neighbourhood of a Legendrian submanifold is naturally convex. Such a convex hypersurface is said to be tubular. On the other hand, the Donaldson-Auroux approximately holomorphic geometry is the main source of non-tubular convex hypersurfaces. For example it provides a pagewise exact symplectic open-book described in §2, and a pair of pages forms a convex hypersurface. (Of course, it is tubular if the page is a cotangent bundle.) Then Theorem 1.2 suggests that a convex hypersurface theory could embody some affinity between high dimensional contact topology and 3-dimensional one.

Further as is described in §1, convex surfaces are objects in Honda's category theory. In [7], the author is trying to generalize this theory to higher dimension. His aim is to show that some quotient of higher dimensional contact category becomes equivalent to Honda's category.

In summary,

QUESTION 3.2. Can we "split" the $2n+3$-dimensional topology of almost contact confoliation into $2n+2$-dimensional (not $2n$-dimensional !) symplectic geometry and 3-dimensional confoliation theory ?

References

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Osaka City University Advanced Mathematical Institute
3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
E-mail: ka-mori@ares.eonet.ne.jp