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On the homeomorphism and diffeomorphism groups fixing a point

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1. Introduction

Let M be a topological metrizable manifold and let $\mathcal{H}(M)$ be the identity component of the group of all compactly supported homeomorphisms of M. By $\mathcal{H}(M,p)$, where $p \in M$, we denote the identity component of the group of all $h \in \mathcal{H}(M)$ with $h(p) = p$.

Definition 1.1. A group G is called *perfect* if it is equal to its own commutator subgroup $[G,G]$, that is $H_1(G) = 0$.

Definition 1.2. A manifold M admits a *compact exhaustion* iff there is a sequence $\{M_i\}_{i=1}^{\infty}$ of compact submanifolds with boundary such that $M_1 \subset \mathrm{Int}M_2 \subset M_2 \subset \dots$ and $M = \bigcup_{i=1}^{\infty} M_i$.

Theorem 1.3. *[3] Assume that either M is compact (possibly with boundary), or M is noncompact boundaryless and admits a compact exhaustion. Then $\mathcal{H}(M)$ is perfect.*

The proof of Theorem 1.3 is a consequence of J.N.Mather's paper combined with results of R.D.Edwards and R.C.Kirby. A special case of Theorem 1.3 was already showed by G.M.Fisher.

2. Main results

Definition 2.1. A group is called *bounded* if it is bounded with respect to any bi-invariant metric.

Definition 2.2. For $g \in [G,G]$ the least k such that g is a product of k commutators is called the *commutator length* of g and is denoted by $\mathrm{cl}_G(g)$. For any perfect group G denote by cld_G the commutator length diameter of G, i.e. $\mathrm{cld}_G := \sup_{g\in G}\mathrm{cl}_G(g)$.

Definition 2.3. A group G is called *uniformly perfect* if G is perfect and $\mathrm{cld}_G < \infty$.

Definition 2.4. Let G be a group. A *conjugation-invariant norm* on G is a function $\nu : G \to [0, \infty)$ for every $g, h \in G$ we have

1. $\nu(g) > 0$ if and only if $g \neq e$,
2. $\nu(g^{-1}) = \nu(g)$,
3. $\nu(gh) \leqslant \nu(g) + \nu(h)$,
4. $\nu(hgh^{-1}) = \nu(g)$.

It is easy to see that G is bounded if and only if any conjugation-invariant norm on G is bounded.

Observe that the commutator length cl_G is a conjugation-invariant norm on $[G, G]$, or on G if G is a perfect group.

Proposition 2.5. *Let G be perfect and bounded group. Then G is uniformly perfect.*

Our main results are the following

Theorem 2.6. *1. The groups $\mathcal{H}(\mathbb{R}^n, 0)$ and $\mathcal{H}(\mathbb{R}^n_+, 0)$ are perfect, where $\mathbb{R}^n_+ = [0, \infty) \times \mathbb{R}^{n-1}$.*

2. Assume that either M is compact (possibly with boundary), or M is noncompact boundaryless and admits a compact exhaustion. Then the group $\mathcal{H}(M, p)$ is perfect.

A similar result was obtained by T.Tsuboi. He proved that $\mathcal{H}([0, 1])$ is perfect by using different argument than that for Theorem 2.6. Next he generalized the result for Lipschitz homeomorphisms and for C^1-diffeomorphisms (resp. C^∞-diffeomorphisms) tangent (resp. infinitely tangent) to the identity at the endpoints. Observe as well that Theorem 2.6 was proved for M closed by K.Fukui in [2]. However, our proof is different than that in [2] and it leads to following theorem.

Theorem 2.7. *The group $\mathcal{H}(\mathbb{R}^n, 0)$ is uniformly perfect and its commutator length diameter is less or equal* 2. *The same is true for $\mathcal{H}(\mathbb{R}^n_+, 0)$.*

Let $\mathcal{D}^r(M)$ (resp. $\mathcal{D}^r(M, p)$) be the identity component of the group of all compactly supported C^r-diffeomorphisms of M (resp. fixing $p \in M$). It is easy to see that $\mathcal{D}^r(M, p)$ is not perfect for $r \geqslant 1$. Moreover, K.Fukui calculated that $H_1(\mathcal{D}^\infty(\mathbb{R}^n, 0)) = \mathbb{R}$.

Theorem 2.8. *1. $\mathcal{H}(\mathbb{R}^n, 0)$ is bounded group.*

2. Assume that either M is compact (possibly with boundary), or M is noncompact boundaryless and admits a compact exhaustion. Then the *group $\mathcal{H}(M)$ is bounded whenever $\mathcal{H}(M,p)$ is bounded.*

Note that this theorem is no longer true in the C^r category for $r \geqslant 1$. Choose a chart at p. Then there is the epimorphism $\mathcal{D}^r(M,p) \ni f \mapsto \mathrm{jac}_p f \in \mathbb{R}_+$, where $\mathrm{jac}_p f$ is the jacobian of f at p in this chart. From Proposition 1.3 in [1] an abelian group is bounded if and only if it is finite and Lemma 1.10 in [1] implies that $\mathcal{D}^r(M,p)$ is unbounded.

3. Questions

Question 3.1. (1) Let $\mathcal{H}^{Lip}(M,p)$ be the compactly supported identity component of Lipschitz homeomorphism group fixing point p. The question is, whether the group $\mathcal{H}^{Lip}(M,p)$ is perfect or bounded.

(2) Denote by $\mathrm{Symp}(M,\omega;p)$ the compactly supported identity component of symplectomorphism group fixing point p. The problem is to calculate $H_1(\mathrm{Symp}(M,\omega;p))$.

(3) The same questions could be asked for contactomorphism groups and volume preserving diffeomorphism groups.

References

[1] D.Burago, S.Ivanov and L.Polterovich, *Conjugation invariant norms on groups of geometric origin*, Advanced Studies in Pures Math. **52**, Groups of Diffeomorphisms (2008), 221–250.

[2] K.Fukui, *Commutators of foliation preserving homeomorphisms for certain compact foliations*, Publ. RIMS, Kyoto Univ. **34-1** (1998), 65–73.

[3] A.Kowalik, T.Rybicki, *On the homeomorphism groups of manifolds and their universal coverings*, Central European Journal of Mathematics vol. **9** (2011), 1217–1231.

[4] J.Lech, I.Michalik, *On the structure of the homeomorphism and diffeomorphism groups fxing a point*, Publicationes Mathematicae Debrecen (in print).

[5] T.Tsuboi, *On the perfectness of groups of diffeomorphisms of the interval tangent to the identity at the endpoints*, Foliations: geometry and dynamics, Warsaw 2000 (ed. by P. Walczak et al.), World Scientific, Singapore (2002), 421–440.

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