Geometry and Foliations 2013
Komaba, Tokyo, Japan

Configuration spaces of linkages on Riemannian surfaces

Mickaël KOURGANOFF

1. Introduction

A mechanical linkage is a mechanism made of rigid rods linked together by flexible joints, in which some vertices are fixed and others may move. The configuration space of a linkage is the set of all its possible positions.

There has been a lot of work on mechanical linkages. Many papers deal with linkages on the Euclidean plane $\mathbb{R}^2$, but the Definition of linkages extends naturally to any Riemannian manifold.

On the Euclidean plane, Kempe [Kem75] has shown in 1875 that for any algebraic curve $\mathcal{C}$, for any euclidian ball $\mathcal{B} \subseteq \mathbb{R}^2$, there exists a linkage $\mathcal{L}$, and one vertex of this linkage v such that $\mathcal{C} \cap \mathcal{B}$ is exactly the set of the possible positions of v (his proof was flawed, but there is a rather simple way to make it correct, see [Abb08]). In particular, the famous *Peaucellier-Lipkin straight-line motion linkage* (Figure 1) forces a vertex to move on a straight line.

More recently, Kapovich and Millson [KM02] have shown that for any smooth compact manifold without boundary M, there exists a linkage for which the configuration space is diffeomorphic to a finite disjoint union of copies of M. Jordan and Steiner proved a weaker version of this theorem with more elementary techniques [JS99]. Thurston already gave lectures on a similar theorem in the 1980's but never wrote a proof.

When we consider the same linkage on two different Riemannian surfaces, for example on the Euclidean plane and on the sphere, the configuration space may be very different. Therefore, it is natural to ask what the two results above become on surfaces other than the plane. Is there a way of characterizing the curves which may be drawn ? May any smooth compact manifold be seen as the configuration space of some linkage ? As far as we know, it is an open problem whether it is possible in general to force a vertex to move on a geodesic. On the sphere, the answer is easy : just take a fixed vertex linked to a moving vertex by an edge of length $\pi/2$. Some solutions also exist on the hyperbolic plane or on the Minkowski plane.

The analogue of the second result in $\mathbb{RP}^2$ (with the metric induced by the natural covering $\mathbb{S}^2 \longrightarrow \mathbb{RP}^2$) is shown in [KM02] using the methods of Mnëv [Mnë88]. The result also applies to $\mathbb{S}^2$ with slight modifications.

However, it seems difficult to show the analogue of Kempe's theorem on the sphere with these methods, because the projective point of view does not distinguish opposite points on the sphere.

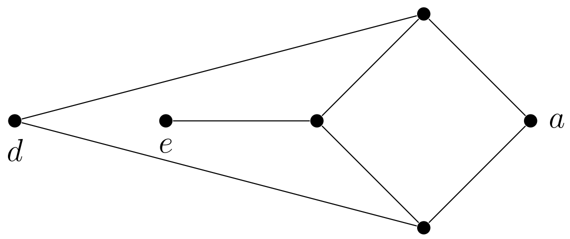

Figure 1: On the plane, the *Peaucellier-Lipkin straight-line motion linkage* forces the point a to move on a straight line. The vertices d and e are fixed to the plane.

2. Main results

DEFINITION 2.1. An *abstract linkage* $\mathcal{L}$ on a Riemannian manifold $\mathcal{N}$ is a graph (V, E) together with :

1. A function $l : E \longrightarrow \mathbb{R}^+$ (which gives the length of each edge) ;
2. A subset $F \subseteq V$ of *fixed vertices* ;
3. A function $\phi_0 : F \longrightarrow \mathcal{N}$ which indicates where the edges of F are fixed.

DEFINITION 2.2. Let $\mathcal{L}$ be an abstract linkage on a manifold $\mathcal{N}$. Let $\mathcal{M}$ be a manifold containing $\mathcal{N}$. A *realization* of a linkage $\mathcal{L}$ on $\mathcal{M}$ is a function $\phi : V \longrightarrow \mathcal{M}$ such that :

1. $\phi|_F = \phi_0$;
2. For each edge $v_1v_2 \in E$, $\delta(\phi(v_1), \phi(v_2)) = l(v_1v_2)$, where δ is the Riemannian distance on $\mathcal{M}$.

DEFINITION 2.3. Let $\mathcal{L}$ be an abstract linkage on a manifold $\mathcal{N}$. Let $W \subseteq V$. Let $\mathcal{M}$ be a manifold containing $\mathcal{N}$. The *partial configuration space of $\mathcal{L}$ on $\mathcal{M}$ with respect to W*, written $\mathcal{E}(W, \mathcal{M})$, is the following set of functions from W to $\mathcal{M}$:

$$\mathcal{E}(W, \mathcal{M}) = \{\phi|_W \mid \phi \text{ realization of } \mathcal{L}\} .$$

DEFINITION 2.4. A *semi-algebraic subset of* $(\mathbb{S}^d)^n$ is a set $A \subseteq (\mathbb{S}^d)^n$ such that there exist $N \geq n$, $m \in \mathbb{N}$ and $f : (\mathbb{R}^{d+1})^N = \mathbb{R}^{(d+1)N} \longrightarrow \mathbb{R}^m$ a polynomial such that :

$$A = \left\{ a \in (\mathbb{S}^d)^n \;\middle|\; \exists b \in (\mathbb{S}^d)^{N-n}, f(a,b) = 0 \right\}.$$

A is called an *algebraic subset of* $(\mathbb{S}^d)^n$ when we can choose $N = n$.

In other words, the semi-algebraic subsets of $(\mathbb{S}^d)^n$ are the projections of the algebraic subsets of $(\mathbb{S}^d)^N$ for any $N \geq n$.

Using techniques similar to [KM02], but with different elementary linkages, we proved :

Theorem 2.5 (Kempe's theorem on $\mathbb{S}^d$, $d \geq 2$)**.** *Let $d \geq 2$ and $n \geq 1$. Let A be a semi-algebraic subset of $(\mathbb{S}^d)^n$. Then there exists an abstract linkage $\mathcal{L} = (V, E, l, F, \phi_0)$ and $W \subseteq V$ such that $\mathcal{E}(W, \mathbb{S}^d) = A$.*

Note that for any linkage $\mathcal{L}$, and for any $W \subseteq V$, $\mathcal{E}(W)$ is a semi-algebraic subset of $(\mathbb{S}^d)^n$, so this theorem describes exactly the sets which are partial configuration spaces.

Kempe's original theorem on the plane was only for $n = 1$ and for algebraic subsets of $\mathbb{R}^2$ intersected with an euclidian ball. However, the corresponding theorem for linkages on $\mathbb{R}^2$, semi-algebraic subsets of $\mathbb{R}^2$ and $n \geq 1$ is a direct consequence of Kapovich and Millson's results [KM02]. A similar theorem for linkages in $\mathbb{R}^d$ has been proved by Timothy Good Abbott [Abb08].

Our proof for Theorem 2.5 also gives a new proof for the following result :

Theorem 2.6 (Differential universality theorem on $\mathbb{S}^d$, $d \geq 2$)**.** *Let M be a smooth compact manifold. There exists a linkage $\mathcal{L}$ on $\mathbb{S}^d$ for which $\mathcal{E}(V, \mathbb{S}^d)$ is diffeomorphic the disjoint union of a finite number of copies of M.*

3. Questions

Question 3.1. Is it possible to replace "*a finite number of copies*" by "*one copy*" in Theorem 2.6 ? (This question is also open on the plane.)

Question 3.2. What happens when $\mathbb{S}^d$ is replaced by any Riemannian manifold in Theorems 2.5 and 2.6 ?

References

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ENS Lyon
46 allée d'Italie, 69364 Lyon Cedex 07, France
E-mail: mickael@kourganoff.fr