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Minimal C^1-diffeomorphisms of the circle which admit measurable fundamental domains

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1. Abstract

This is a joint work with Shigenori Matsumoto (Nihon University).

The concept of ergodicity is important not only for measure preserving dynamical systems but also for systems which admits a natural quasi-invariant measure. Given a probability space (X, μ) and a transformation T of X, μ is said to be *quasi-invariant* if the push forward $T_*\mu$ is equivalent to μ. In this case T is called *ergodic* with respect to μ, if a T-invariant Borel subset in X is either null or conull.

A diffeomorphism of a differentiable manifold always leaves the Riemannian volume (also called the Lebesgue measure) quasi-invariant, and one can ask if a given diffeomorphism is ergodic with respect to the Lebesgue measure (below *ergodic* for short) or not. Answering a question of A. Denjoy [D], A. Katok (see for instance Chapt. 12.7, p. 419, [KH]), and independently M. Herman (Chapt. VII, p. 86, [H]) showed the following theorem.

Theorem 1.1. *A C^1-diffeomorphism of the circle with derivative of bounded variation is ergodic provided its rotation number is irrational.*

At the opposite extreme of the ergodicity lies the concept of measurable fundamental domains. Given a transformation T of a standard probability space (X, μ) leaving μ quasi-invariant, a Borel subset C of X is called a *measurable fundamental domain* if T^nC $(n \in \mathbb{Z})$ is mutually disjoint and the union $\cup_{n\in\mathbb{Z}} T^nC$ is conull. In this case any Borel function on C can be extended to a T-invariant measurable function on X, and an ergodic component of T is just a single orbit. The purpose of this talk is to show the following theorem.

Theorem 1.2 ([KM]). *For any irrational number α, there is a minimal C^1-diffeomorphism of the circle with rotation number α which admits a measurable fundamental domain with respect to the Lebesgue measure.*

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To prove the theorem, first we construct a Lipschitz homeomorphism F with rotation number α which admits a measurable fundamental domain. We regard the circle S^1 as $\mathbb{R}/\mathbb{Z}$. Suppose R denotes the rotation by α.

Claim 1.3. *For any irrational number α, we can construct a Cantor set $C \in S^1$ so that $R^n C \cap R^m C = \emptyset$ for any integers $n \neq m$.*

Admitting this claim, fix a probability measure μ_0 on C without atom such that $\operatorname{supp}(\mu_0) = C$. We also choose a sequence $(a_i)_{i\in\mathbb{Z}}$ of positive numbers satisfying $\sum_{i\in\mathbb{Z}} a_i = 1$. Now we can define a probability measure μ on S^1 by

$$\mu := \sum_{i\in\mathbb{Z}} a_i R^i_* \mu_0. \tag{1.4}$$

The Radon-Nikodym derivative $\frac{dR^{-1}_*\mu}{d\mu}$ is equal to $\frac{a_{i+1}}{a_i}$ on the set $R^i C$. Now we assume that $\frac{a_{i+1}}{a_i} \in [\frac{1}{D}, D]$ for some $D > 1$, then it follows that $\frac{dR^{-1}_*\mu}{d\mu} \in L^\infty(S^1, \mu)$.

We define a homeomorphism h of S^1 by $h(0) = 0$ and $h(x) = y$ if and only if $Leb[0, x] = \mu[0, y]$, where Leb denotes the Lebesgue measure on S^1; or more briefly, $h_*\, Leb = \mu$. Finally define a homeomorphism F of S^1 by $F := h^{-1} \circ R \circ h$, then

$$\frac{dF^{-1}_*\, Leb}{d\, Leb} = \frac{dR^{-1}_*\mu}{d\mu} \circ h \in L^\infty(S^1, Leb), \tag{1.5}$$

i.e. the map F is a Lipschitz homeomorphism. The set $C' = h^{-1}C$ is a measurable fundamental domain of F.

To prove Theorem 1.2, it is enough to make the Radon-Nikodym derivative $g = \frac{dR^{-1}_*\mu}{d\mu}$ continuous on S^1. Assume that g is continuous, set $\phi = \log g$ and

$$\begin{aligned} \phi^{(m)}(x) &= \sum_{i=0}^{m-1} \phi(R^i x) \qquad (m > 0), \\ \phi^{(-m)}(x) &= -\sum_{i=1}^{m} \phi(R^{-i} x) \qquad (m > 0), \\ \phi^{(0)}(x) &= 0, \end{aligned} \tag{1.6}$$

then we can conclude that $a_i = \exp(\phi^{(i)}(x_0))a_0$ for any point $x_0 \in C$. Since $\sum_{i\in\mathbb{Z}} a_i = 1$, the sum $\sum_{i\in\mathbb{Z}} \exp(\phi^{(i)}(x_0))$ has to be finite.

Fix an integer $n \in \mathbb{N}$. Since $R^{-2^n}C, \dots, C, \dots, R^{2^n-1}C$ are disjoint compact sets, for a sufficiently small ε-neighbourhood N of C, $R^{-2^n}N, \dots,$

$N, \dots, R^{2^n-1}N$ are also disjoint. Take a bump function $f\colon S^1 \to \mathbb{R}$ so that $\operatorname{supp} f \subset N$, $f(x) = (3/4)^n$ for $x \in C$ and $0 \le f(x) < (3/4)^n$ for $x \in N \setminus C$. Define $\phi_n\colon S^1 \to \mathbb{R}$ by

$$\phi_n(x) = \begin{cases} -f(R^{-i}x) & x \in R^iN,\ i = 0, 1, \dots, 2^n - 1 \\ f(R^{-i}x) & x \in R^iN,\ i = -2^n, -2^n + 1, \dots, -1 \\ 0 & \text{otherwise} \end{cases} \tag{1.7}$$

and $\phi = \sum_{i=1}^{\infty} \phi_n$, then ϕ is a continuous function satisfying

$$\sum_{i \in \mathbb{Z}} \exp(\phi^{(i)}(x_0)) < \infty. \tag{1.8}$$

Employing this ϕ, set $\tilde{\mu} = \sum_{i \in \mathbb{Z}} (\exp \circ \phi^{(i)} \circ R^{-i}) R^i_* \mu_0$ and $\mu = \frac{\tilde{\mu}}{\int_{S^1} d\tilde{\mu}}$. The function $F\colon S^1 \to S^1$ constructed from this μ is C^1.

References

[D] A. Denjoy, *Sur les courbes défini par les équations différentielle à la surfase du tore.* J. Math. Pures Appl. 9(11) (1932), 333-375.

[H] M. R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle a des rotations.* Publ. Math. I. H. E. S., 49(1979), 5-242.

[KH] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems.* Encyclopedia of Mathematics and its Applications, Vol. 54, Cambridge University Press, 1995.

[KM] H. Kodama and S. Matsumoto, *Minimal C^1-diffeomorphisms of the circle which admit measurable fundamental domains.* Proc. Amer. Math. Soc. 141 (2013), 2061-2067, `arXiv:1005.0585v2`

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