Geometry and Foliations 2013
Komaba, Tokyo, Japan

Superheavy subsets and noncontractible Hamiltonian circle actions

Morimichi KAWASAKI

1. Introduction

Let (M,ω) be a symplectic manifold. In this paper a diffeomorphism f of M is called *a symplectomorphism* if f preserves the symplectic form ω.

Our result is as follows:

Theorem 1.1. *Let $(\mathbb{T}^2,\omega_{\mathbb{T}^2}) = (\mathbb{R}^2/\mathbb{Z}^2,\omega_{\mathbb{T}^2})$ be the 2-torus with the coordinates (p,q) and the symplectic form $dp\wedge dq$. The union $M\cup L$ of the meridian curve M and the longitude curve L is a "$[\mathbb{T}^2]$-superheavy" subset of $(\mathbb{T}^2,\omega_{\mathbb{T}^2})$.*

As a corollary of Proposition 1.1, we have the following result:

Corollary 1.2. *Let $(\mathbb{C}P^n,\omega_{FS})$ be the complex projective space with the Fubini-Study form ω_{FS} and C be the Clifford torus $\{[z_0 : \cdots : z_n] \in \mathbb{C}P^n; |z_0| = \cdots = |z_n|\}$ of $\mathbb{C}P^n$. Then there exists no symplectomorphism f of $(\mathbb{C}P^n \times \mathbb{T}^2, \omega_{FS} \oplus \omega_{\mathbb{T}^2})$ such that $C{\times}(M\cup L) \cap f(C{\times}(M\cup L)) = \emptyset$.*

2. Preliminaries

2.1. Definitions

For a function $F\colon M \to \mathbb{R}$ with compact support, we define *the Hamiltonian vector field* $\operatorname{sgrad} F$ associated with F by

$$\omega(\operatorname{sgrad} F, V) = -dF(V) \text{ for any } V \in \mathcal{X}(M),$$

where $\mathcal{X}(M)$ denotes the set of smooth vector fields on M.

For a function $F\colon M{\times}[0,1] \to \mathbb{R}$ and $t \in [0,1]$, we define $F_t\colon M \to \mathbb{R}$ by $F_t(x) = F(x,t)$. We denote by $\{f_t\}$ the isotopy which satisfies $f_0 = \mathrm{id}$ and $\frac{d}{dt}f_t(x) = (\operatorname{sgrad} F_t)_{f_t(x)}$. We call this *the Hamiltonian path generated by the Hamiltonian function F*. The time-1 map f_1 of $\{f_t\}$ is called *the Hamiltonian diffeomorphism generated by the Hamiltonian function F*. A diffeomorphism f is called *a Hamiltonian diffeomorphism* if there exists a

Hamiltonian function with compact support generating f. A Hamiltonian diffeomorphism is a symplectomorphism.

For a symplectic manifold (M, ω), we denote by $\mathrm{Symp}(M, \omega)$, $\mathrm{Ham}(M, \omega)$ and $\widetilde{\mathrm{Ham}}(M, \omega)$, the group of symplectomorphisms, the group of Hamiltonian diffeomorphisms of (M, ω) and its universal cover, respectively. We denote by $\mathrm{Symp}_0(M, \omega)$ the identity component of $\mathrm{Symp}(M, \omega)$. Note that $\mathrm{Ham}(M, \omega)$ is a normal subgroup of $\mathrm{Symp}_0(M, \omega)$.

DEFINITION 2.1. For functions F and G and a symplectic manifold (M, ω), *the Poisson bracket* $\{F, G\} \in C^\infty(M)$ is defined by

$$\{F, G\} = \omega(\mathrm{sgrad}\, G, \mathrm{sgrad}\, F).$$

DEFINITION 2.2 ([1]). Let (M, ω) be a symplectic manifold.

A subset U of M is called *displaceable* if there exists a Hamiltonian diffeomorphism $f \in \mathrm{Ham}(M, \omega)$ such that $f(U) \cap \bar{U} = \emptyset$.

A subset U of M is called *strongly displaceable* if there exist a symplectomorphism $f \in \mathrm{Symp}(M, \omega)$ such that $f(U) \cap \bar{U} = \emptyset$.

We consider the cotangent bundle $T^*\mathbb{S}^1 = \mathbb{R} \times \mathbb{S}^1$ of the circle $\mathbb{S}^1$ with the coordinates (r, θ) and the symplectic form $dr \wedge d\theta$. A subset U of M is called *stably displaceable* if $U \times \{r = 0\}$ is displaceable in $M \times T^*\mathbb{S}^1$ equipped with the split symplectic form $\bar{\omega} = \omega \oplus (dr \wedge d\theta)$.

If U is displaceable, then U is stably displaceable. Since $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega)$, if U is displaceable, then U is strongly displaceable.

2.2. Spectral invariants

For a closed connected symplectic manifold (M, ω), define

$$\Gamma = \frac{\pi_2(M)}{\mathrm{Ker}(c_1) \cap \mathrm{Ker}([\omega])},$$

where c_1 is the first Chern class of TM with an almost complex structure compatible with ω. The Novikov ring of the closed symplectic manifold (M, ω) is defined as follows:

$$\Lambda = \left\{ \sum_{A \in \Gamma} a_A A; a_A \in \mathbb{Q}, \#\{A; a_A \neq 0, \int_M \omega < R\} < \infty \text{ for any real number } R \right\}$$

The quantum homology $QH_*(M, \omega)$ is a Λ-module isomorphic to $H_*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda$ and $QH_*(M, \omega)$ has a ring structure with the multiplication called *quantum product* [3]. To each element $a \in QH_*(M, \omega)$, a functional $c(a, \cdot)\colon C^\infty(M \times [0, 1]) \to \mathbb{R}$ is defined in terms of Hamiltonian Floer theory. The functional

$c(a,\cdot)$ is called *spectral invariant* ([3]). To describe the properties of a spectral invariant, we define the spectrum of a Hamiltonian function as follows:

DEFINITION 2.3 ([3]). Let $H \in C^\infty(M\times[0,1])$ be a Hamiltonian function on a closed symplectic manifold M. *Spectrum* $\mathrm{Spec}(H)$ of H is defined as follows:

$$\mathrm{Spec}(H) = \left\{ \int_0^1 H(h_t(x),t)dt + \int_{\mathbb{D}^2} u^*\omega \right\} \subset \mathbb{R},$$

where $\{h_t\}_{t\in[0,1]}$ is the Hamiltonian path generated by H and $x \in M$ is a fixed point of h_1 whose orbit defined by $\gamma^x(t) = h_t(x)$ $(t \in [0,1])$ is a contractible loop and $u\colon \mathbb{D}^2 \to M$ is a disc in M such that $u|_{\partial\mathbb{D}^2} = \gamma^x$.

We define the *non-degeneracy* of Hamiltonian functions as follows:

DEFINITION 2.4. $H \in C^\infty(M\times[0,1])$ is called *non-degenerate* if the graph of the Hamiltonian diffeomorphism h generated by H is transverse to the diagonal in $M\times M$.

The followings are well-known properties of spectral invariants ([3], [4]).

Non-degenerate spectrality $c(a,H) \in \mathrm{Spec}(H)$ for every non-degenerate $H \in C^\infty(M\times[0,1])$.

Hamiltonian shift property $c(a,H+\lambda(t)) = c(a,H) + \int_0^1 \lambda(t)dt$.

Monotonicity property If $H_1 \leq H_2$, then $c(a,H_1) \leq c(a,H_2)$.

Lipschitz property The map $H \mapsto c(a,H)$ is Lipschitz on $C^\infty(M\times[0,1])$ with respect to the C^0-norm.

Symplectic invariance $c(a,f^*H) = c(a,H)$ for any $f \in \mathrm{Symp}_0(M,\omega)$ and any $H \in C^\infty(M\times[0,1])$.

Homotopy invariance $c(a,H_1) = c(a,H_2)$ for any normalized H_1 and H_2 generating the same $h \in \widetilde{\mathrm{Ham}}(M)$. Thus one can define $c(a,\cdot)\colon \widetilde{\mathrm{Ham}}(M) \to \mathbb{R}$ by $c(a,h) = c(a,H)$, where H is a normalized Hamiltonian function generating h.

Triangle inequality $c(a*b,fg) \leq c(a,f) + c(b,g)$ for elements f and $g \in \widetilde{\mathrm{Ham}}(M,\omega)$, where $*$ denotes the quantum product.

2.3. Heaviness and superheaviness

M. Entov and L. Polterovich ([1]) defined the *heaviness* and the *superheaviness* of closed subsets in closed symplectic manifolds and gave stably non-displaceable subsets and strongly non-displaceable subsets.

For an idempotent a of the quantum homology $QH_*(M,\omega)$, define the functional $\zeta_a\colon C^\infty(M) \to \mathbb{R}$ by

$$\zeta_a(H) = \lim_{l\to\infty} \frac{c(a, lH)}{l},$$

where $c(a,H)$ is the spectral invariant ([3], see Section 2.2).

Definition 2.5 ([1]). Let (M,ω) be a $2n$-dimensional closed symplectic manifold. Take an idempotent a of the quantum homology $QH_*(M,\omega)$.

A closed subset X of M is called *ζ_a-heavy* (or *a-heavy*) if

$$\zeta_a(H) \geq \inf_X H \text{ for any } H \in C^\infty(M),$$

and is called *ζ_a-superheavy* (or *a-superheavy*) if

$$\zeta_a(H) \leq \sup_X H \text{ for any } H \in C^\infty(M).$$

A closed subset X of M is called heavy (respectively, superheavy) if X is ζ_a-heavy (respectively, ζ_a-superheavy) for some idempotent a of $QH_*(M,\omega)$.

For a oriented closed manifold M, we denote its fundamental class by $[M]$.

Theorem 2.6 (A part of Theorem 1.4 of [1]). *For a non-trivial idempotent a of $QH_*(M,\omega)$, the followings hold.*

(1) Every ζ_a-superheavy subset is ζ_a-heavy.

(2) Every ζ_a-heavy subset is stably non-displaceable.

(3) Every $[M]$-superheavy subset is strongly non-displaceable.

□

Example 2.7. (1) Let $(\mathbb{T}^2, \omega_{\mathbb{T}^2}) = (\mathbb{R}^2/\mathbb{Z}^2, \omega_{\mathbb{T}^2})$ be the 2-torus with the coordinates (p,q) and the symplectic form $dp\wedge dq$. Then the meridian curve $M = \{(p,q) \in \mathbb{T}^2; q = 0\}$ and the longitude curve $L = \{(p,q) \in \mathbb{T}^2; p = 0\}$ are $[\mathbb{T}^2]$-heavy subsets of $(\mathbb{T}^2, \omega_{\mathbb{T}^2})$, hence they are stably non-displaceable ([1] Example 1.18).

(2) Let $(\mathbb{C}P^n, \omega_{FS})$ be the complex projective space with the Fubini-Study form. The Clifford torus $C = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n; |z_0| = \cdots = |z_n|\} \subset \mathbb{C}P^n$ is a $[\mathbb{C}P^n]$-superheavy subset of $(\mathbb{C}P^n, \omega_{FS})$, hence they are strongly non-displaceable ([1] Theorem 1.8).

Definition 2.8. Let (M, ω) be a $2n$-dimensional closed symplectic manifold. Take an idempotent a of the quantum homology $QH_*(M, \omega)$. An open subset U of M is said to be ζ_a-null if for $G \in C^\infty(U)$,

$$\zeta_a(G) = 0.$$

An open subset U of M is said to be strongly ζ_a-null if for $F \in C^\infty(M)$ and $G \in C^\infty(U)$ such that $\{F, G\} = 0$,

$$\zeta_a(F + G) = \zeta_a(F).$$

A subset X of M is said to be (strongly) ζ_a-null if there exists a (strongly) ζ_a-null open neighborhood U of X.

3. Main proposition

Definition 3.1. A closed symplectic manifold (M, ω) is called *rational* if $\omega(\pi_2(M))$ is a discrete subgroup of $\mathbb{R}$.

The main result is the following proposition. We use this proposition to prove Theorem 1.1 by using the argument of stems.

Proposition 3.2. *Let (M, ω) be a rational closed symplectic manifold. Let α be a nontrivial free homotopy class of free loops on M; $\alpha \in [\mathbb{S}^1, M]$, $\alpha \neq 0$. Let U be an open subset of M. Assume that there exists a Hamiltonian function $H \in C^\infty(M \times [0,1])$ which satisfies the followings:*

(1) $h_1|_U = \mathrm{id}_U$,

(2) for any $x \in U$, the free loop $\gamma^x \colon \mathbb{S}^1 \to M$ defined by $\gamma^x(t) = h_t(x)$ belongs to α, and

(3) $\alpha \notin i_*([\mathbb{S}^1, U])$.

Here $i \colon U \to M$ is the inclusion map and $\{h_t\}_{t\in[0,1]}$ is the Hamiltonian path generated by H. Then U is strongly ζ_a-null for any idempotent a of $QH_*(M, \omega)$.

□

The proof of Theorem 3.2 is based on the idea of K. Irie in the proof of Theorem 2.4 of [2].

4. Proof of Theorem 1.1

M. Entov and L. Polterovich defined stems to give examples of superheavy subsets. We define ζ_a-stems which generalizes a little the notion of stems and there exhibits ζ_a-superheaviness.

We generalize the argument of Entov and Polterovich as follows.

Definition 4.1. Let $\mathbb{A}$ be a finite-dimensional Poisson-commutative subspace of $C^\infty(M)$ and $\Phi\colon M \to \mathbb{A}^*$ be the moment map defined by $\langle \Phi(x), F\rangle = F(x)$. Let a be a non-trivial idempotent of $QH_*(M,\omega)$. A non-empty fiber $\Phi^{-1}(p)$, $p \in \mathbb{A}^*$ is called *a ζ_a-stem* of $\mathbb{A}$ if all non-empty fibers $\Phi^{-1}(q)$ with $q \neq p$ is strongly ζ_a-null. If a subset of M is a ζ_a-stem of a finite-dimensional Poisson-commutative subspace of $C^\infty(M)$, it is called just *a ζ_a-stem*.

Theorem 4.2. *For every idempotent a of $QH_*(M,\omega)$, every ζ_a-stem is a ζ_a-superheavy subset.*

□

Proof of Theorem 1.1.
Note that $(\mathbb{T}^2, \omega_{\mathbb{T}^2})$ is rational. Consider a momentum map $\Phi \in C^\infty(\mathbb{T}^2)$ such that $\Phi(x) = 0$ if $x \in M \cup L$ and $\Phi(x) > 0$ if $x \notin M \cup L$. Take a real number $\epsilon \neq 0$. Then there exist a positive number δ and an open neighborhood U of $\Phi^{-1}(\epsilon)$ such that $U \subset (\delta, 1-\delta) \times (\delta, 1-\delta)$. Consider a Hamiltonian function $H \in C^\infty(\mathbb{T}^2 \times [0,1])$ such that $H((p,q),t) = p$ for any $p \in [\delta, 1-\delta]$.

Define the free loop $\gamma\colon \mathbb{S}^1 \to \mathbb{T}^2$ by $\gamma(t) = (0,t)$. Let $\alpha \in [\mathbb{S}^1, \mathbb{T}^2]$ be the homotopy class of free loops represented by γ. Then α, U and H satisfy the assumptions of Theorem 3.2, hence U satisfies is strongly ζ_a-null. Thus $M \cup L$ is a ζ_a-stem, hence it is ζ_a-superheavy.

■

5. Proof of Corollary 1.2

We use the following theorem to prove Corollary 1.2.

Theorem 5.1 ([1] Theorem 1.7)**.** *Let (M_1,ω_1) and (M_2,ω_2) be closed symplectic manifolds. Take non-zero idempotents a_1, a_2 of $QH_*(M_1)$, $QH_*(M_1)$, respectively. Assume that for $i = 1,2$, X_i be a a_i-heavy (respectively, a_i-superheavy) subset. Then the product $X_1 \times X_2$ is $a_1 \otimes a_2$-heavy (respectively, $a_1 \otimes a_2$-superheavy) subset of $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ of $QH_*(M_1 \times M_2)$.*

□

Proof of Corollary 1.2. By Example 2.7 and Theorem 1.1, Theorem 5.1, $C \times (M \cup L)$ is $[C \times (M \cup L)]$-superheavy subset of $(\mathbb{C}P^n \times \mathbb{T}^2, \omega_{FS} \oplus \omega_{\mathbb{T}^2})$. Thus Theorem 2.6 implies that there exists no symplectomorphism f such that $C \times (M \cup L) \cap f(C \times (M \cup L)) = \emptyset$.

■

References

[1] M. Entov and L. Polterovich, Rigid subsets of symplectic manifolds, *Comp. Math.* **140** (2009), 773–826.

[2] K. Irie, Hofer-Zehnder capacity and a Hamiltonian circle action with noncontractible orbits, arXiv:1112.5247v1.

[3] Y. -G. Oh, Floer mini-max theory, the Cerf diagram, and the spectral invariants, *J. Korean Math. Soc.* **46** (2009), 363–447.

[4] M. Usher, Spectral numbers in Floer theories, *Compos. Math.* **144**(6) (2008), 1581–1592.

Graduate School of Mathematical Sciences, The University of Tokyo.
3-8-1 Komaba Meguro-ku Tokyo 153-8914, Japan
E-mail: cawasaki@ms.u-tokyo.ac.jp