Geometry and Foliations 2013
Komaba, Tokyo, Japan

Lie foliations transversely modeled on nilpotent Lie algebras

Naoki KATO

1. Introduction

Let M be an n-dimensional closed orientable smooth manifold and let $\mathcal{F}$ be a codimension q transversely orientable smooth foliation of M. Let $\mathfrak{g}$ be a q-dimensional Lie algebra over $\mathbb{R}$.

Definition 1.1. The foliation $\mathcal{F}$ is a Lie $\mathfrak{g}$-foliation if there exists a non-singular Maurer-Cartan form $\omega \in A^1(M, \mathfrak{g})$ such that $T\mathcal{F} = \mathrm{Ker}(\omega)$.

P. Molino [4] proved that the following structure theorem.

Theorem 1.2 (Molino)**.**

1. *There exists a locally trivial fibration $\pi\colon M \to W$ such that each fiber is the closure of a leaf of $\mathcal{F}$.*
2. *There exists a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is uniquely determined by $\mathcal{F}$ such that, for each fiber F of the fibration π, the induced foliation $\mathcal{F}|_F$ is a Lie $\mathfrak{h}$-foliation.*

The Lie algebra $\mathfrak{h}$ is called the structure Lie algebra of $\mathcal{F}$.

By Theorem 1.2, to each Lie foliation $\mathcal{F}$, there are associated two Lie algebras, the model Lie algebra $\mathfrak{g}$ and the structure Lie algebra $\mathfrak{h}$. Hence, we have a natural question to determine the pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ which can be realized as a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of a closed manifold M with structure Lie algebra $\mathfrak{h}$.

Definition 1.3. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. $(\mathfrak{g}, \mathfrak{h})$ is realizable if there exists a closed manifold M and a Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of M such that the structure Lie algebra of $\mathcal{F}$ is $\mathfrak{h}$.

If $\mathcal{F}$ is a flow, then the structure Lie algebra $\mathfrak{h}$ is abelian and thus it is isomorphic to $\mathbb{R}^m$ for some m.

Definition 1.4. Let $\mathfrak{g}$ be a Lie algebra and m be an integer. $(\mathfrak{g}, m)$ is realizable if there exists a closed manifold M and a Lie $\mathfrak{g}$-flow $\mathcal{F}$ of M such that the structure Lie algebra of $\mathcal{F}$ is $\mathbb{R}^m$, that is the dimension of the structure Lie algebra is equal to m.

E. Gallego, B. Herrera, M. Llabrés and A. Reventós completely solved this problem in the case where the dimension of the Lie algebras $\mathfrak{g}$ is three (cf. [2], [3]).

We study the realizing problems of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}, m)$ in the case where $\mathfrak{g}$ is nilpotent Lie algebras of general dimensions.

2. Main results

Theorem 2.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra which has a rational structure. Then $(\mathfrak{g}, m)$ is realizable if and only if $m \leq \dim \mathfrak{c}(\mathfrak{g})$, where $\mathfrak{c}(\mathfrak{g})$ is the center of $\mathfrak{g}$.*

Theorem 2.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{h})$ is realizable if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and the quotient Lie algebra $\mathfrak{h}\backslash\mathfrak{g}$ has a rational structure.*

Corollary 2.3. *For any nilpotent Lie algebra $\mathfrak{g}$, there exists a minimal Lie $\mathfrak{g}$-foliation $\mathcal{F}$ of a closed manifold M.*

Since nilpotent Lie algebras has a non-trivial center, by Theorem 2.1, any nilpotent Lie algebra $\mathfrak{g}$ with a rational structure can be realized as a Lie $\mathfrak{g}$-flow. On the other hand, there exists a nilpotent Lie algebra $\mathfrak{g}$ with no rational structures which can not be realized as a Lie $\mathfrak{g}$-flow.

Example 2.4 (Chao). Let $c_{ij}^k, 1 \leq i, j \leq m, 1, \leq k \leq n$ be real numbers such that $c_{ij}^k = -c_{ji}^k$. Assume that c_{ij}^k are algebraically independent over $\mathbb{Q}$. Let $\mathfrak{g}$ be the Lie algebra defined by a basis

$$\{X_1, \ldots, X_m, Y_1, \ldots, Y_n\}$$

with the products

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k Y_k$$

for $i, j = 1, \ldots, m$ and all other products being zero. Then $\mathfrak{g}$ is nilpotent a Lie algebra and $[\mathfrak{g}, \mathfrak{g}] = \langle Y_1, \ldots, Y_n \rangle_{\mathbb{R}}$. This Lie algebra $\mathfrak{g}$ has no rational structure if $(n/2)(m^2 - m) > m^2 + n^2$.

Proposition 2.5. *Let $\mathfrak{g}$ be the Lie algebra constructed above. If $(n/2)(m^2 - m) > (m+1)^2 + (n+1)^2$, then $\mathfrak{g}$ can not be realized as a Lie $\mathfrak{g}$-flow.*

However there exists a nilpotent Lie algebra with no rational structures which can be realized as a Lie flow.

Proposition 2.6. *There exists a nilpotent Lie algebra* $\mathfrak{g}$ *which has no rational structures such that* $\mathfrak{g}$ *can be realized as a Lie* $\mathfrak{g}$*-flow.*

References

[1] C. Y. Chao, *Uncountably many nomisomorphic nilpotent Lie algebras*, Proc. Amer. Math. Soc. **13** (1962), 903–906.

[2] E. Gallego and A. Reventós, *Lie Flows of Codimension* 3, Trans. Amer. Math. Soc. **326** (1991), 529–541.

[3] B. Herrera, M. Llabrés and A. Reventós, *Transverse structure of Lie foliations*, J. Math. Soc. Japan **48** (1996), 769–795.

[4] P. Molino, *Géométrie globale des feuilletages riemanniens*, Proc. Kon. Nederl. Akad, Ser. A1 **85** (1982), 45–76.

Shibaura Institute of Technology
307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570 JAPAN
E-mail: i035983@sic.shibaura-it.ac.jp