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Classsification of maximal codimension totally geodesic foliations of the complex hyperbolic space

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1. Totally geodesic foliations of $\mathbb{H}^n$

Totally geodesic foliations of the real hyperbolic space $\mathbb{H}^n$ in codimension 1 are well understood. The first classification given by Ferus in [5] concentrates on geometry of orthogonal transversal. Browne observed that it is enough to study vector fields along geodesics (cf. [2]). Lastly, Lee and Yi classified totally geodesic codimension 1 foliations of $\mathbb{H}^n$ through closed curves on S^{n-1} which represent the ideal boundary of leaves. For short explanation compare [4] and [1].

2. Complex hyperbolic space and complex de Sitter space

The complex hyperbolic space $\mathbb{C}H^n$ is one of the easiest examples of the Hadamard manifold with nonconstant sectional curvature. Even here there is no (real) codimesion 1 totally geodesic submanifolds; in fact only totally geodesic submanifolds are totatlly complex or totally real (cf. [6]).

Define *complex de Sitter space* $\mathbb{C}\Lambda^n$ as the (complex) projectivization of positive vectors with respect to the Hermitian form in $\mathbb{C}^{n+1}$ given by

$$\langle Z, W\rangle = -Z_0\overline{W_0} + Z_1\overline{W_1} + \ldots + Z_n\overline{W_n}.$$

Recall that $\mathbb{C}H^n$ is simply projectivization of negative vectors in $\mathbb{C}^{n+1}$.

Every totally geodesic codimension 2 submanifold of $\mathbb{C}H^n$ is the projectivization of complex hyperplane which is complex-time-like. Thus it is represented by a positive vector i.e. belonging to $\mathbb{C}\Lambda^n$.

3. Classsification of totally geodesic codimension 2 foliations of $\mathbb{C}H^n$

In [4] Czarnecki and Walczak stated the problem of geometric classification of foliations of $\mathbb{C}H^n$ with leaves isometric to $\mathbb{C}H^{n-1}$, i.e. of the real codimension 2.

This problem could be studied similarly to the real case when the conformal geometry is applied. Using methods developed in [7] Czarnecki and Langevin (see [3]) gave local and global conformal condition for curves in de Sitter space Λ^{n+2} to represent a totally geodesic codimension 1 folations of $\mathbb{H}^n$.

Totally geodesic codimension 2 foliations are curves in $\mathbb{C}\Lambda^n$ such that its tangent vector is of complex-time-like. Therefore, totally geodesic maximal codimension foliations of $\mathbb{C}H^n$ are those which are orthogonal to a complex curve of holomorphic curvature bounded by 1. Such a curve is an Hadamard 2–dimensional submanifold of bounded negative curvature.

References

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