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Transverse Ricci flow as a tool for classification of Riemannian flows

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1. Introduction

A Riemannian flow is a 1-dimensional Riemannian foliation. It provides just enough structure (an invariant metric on the normal bundle) with just enough flexibility (the flow need not to be isometric) to be both treatable and interesting for geometers. For similar reasons, it is also of interest to people working in dynamical systems.

In 1984 Yves Carrière gave a classification of Riemannian flows on 3-manifolds, [1]. This stemed from his previous work on the topic, but with more emphasis on Molino's structural approach. Indeed, not long after Molino and Almeida classified Riemannian flows on 4-manifolds, [2]. The two theorems can be summarized as folows

Theorem 1.1 (Carrière). *A 3-manifold with a Riemannian flow*

- *is either foliated-diffeomorphic to a dense linear flow on a torus, or*
- *a suspension of a prescribed type, or*
- *foliated-diffeomorphic to a prescribed flow on a torus, or*
- *foliated-diffeomorphic to a prescribed flow on a lens space, or*
- *a Seifert fibration,*

and those instances are distinguished by presence or absence of dense or closed leaves and their holonomy. Only the second dot cannot be endowed with a metric that makes the flow isometric.

Theorem 1.2 (Molino, Almeida). *A 4-manifold with a Riemannian flow*

- *is either foliated-diffeomorphic to a dense linear flow on a torus, or*
- *a suspension of a prescribed type, or*
- *foliated-diffeomorphic to a prescribed flow on a (twisted) double of* $\mathbb{T}^2 \times \mathbb{D}^2$*, or*
- *has a 2-dimensional orbifold with boundary as the space of leaves, or*

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- *is a Seifert fibration,*

and, again, those instances are distinguished by occuring closures of leaves and their holonomy. This time both the second and the fourth dots provide examples of non-isometric flows.

It seems there was no substantial progress in classifying Riemannian flows in higher dimensions and this is not very surprising – this problem essentially generalizes classification of manifolds. We emphasize that both proofs rely on Molino's theory and structural approach presented in [3].

We would like to recover those theorems using different methods, namely geometric flows. In 1986 Min-Oo and Ruh restated Hamilton's result on Ricci flow in terms of flow of Cartan connections, [4]. Recently, with Lovrić, in [5], they were able to apply this technique to a flow of connections on transverse bundle of a foliation, essentially obtaining the Ricci flow on the transverse manifold – with usual consequences:

Theorem 1.3 (Lovrić, Min-Oo, Ruh)**.** *Suppose a codimension 3 Riemannian foliation with positive definite Ricci curvature. Then the metric can be deformed to a metric of constant sectional curvature.*

The metric here is understood only on the transverse bundle. We would like to better comprehend and to apply this transverse Ricci flow.

2. Main results

The present work is concerned with recovering Theorems 1.1 and 1.2 with methods developed in [5].

Proposition 2.1. *Holonomy and stratification that classify codimension 2 and 3 Riemannian flows can be deduced from curvature properties of the deformed metric.*

It is plausible that transverse Ricci flow in those codimensions should prove as useful as it's usual, non-transverse counterpart. Of course, topological considerations threaten to be much more involved (we have metric information only in a normal, non-integrable direction), but on the other hand, we know – a posteriori – that spaces in question are quite simple.

Another step is a geometric proof of the following

Theorem 2.2. *Compact, connected, orientable, irreducible, atoroidal 3-orbifold is geometric.*

This is one possible statement of Thurston's Geometrization Conjecture for orbifolds. Recall that an orbifold is a topological space locally homeomorphic to an euclidean space divided by a finite group action. Original (quite involved) proofs are due to Cooper, Hodgson, Kerckhoff (cf. [6]) and Boileau, Leeb, Porti (cf. [7]) and carefully reduce the problem to Geometrization Conjecture for manifolds. Recently, Kleiner and Lott provided in [8] a direct proof, describing Ricci flow on orbifolds. It still seems a worthwile task to prove Theorem 2.2 using the folowing desingularization procedure (cf. [9])

Theorem 2.3. *Every orbifold can be realized as a space of leaves of a Riemannian foliation. Geometry of the orbifold is the transverse geometry of that foliation.*

and using [5]. Note that this realization may produce a foliation of high dimension, although the codimension is preserved.

Proposition 2.4. *Transverse flow of Cartan connections of [5] and Ricci flow on orbifolds of [8] coincide.*

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