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Contact manifolds with symplectomorphic symplectizations

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1. Introduction

Contact geometry and symplectic geometry are very much related. Given a contact manifold (M, ξ), we can associate a symplectic manifold $(\mathrm{S}_\xi M, \omega_\xi)$, called its *symplectization*. Topologically, the symplectization of M is just the product $\mathbb{R} \times M$. There is an $\mathbb{R}$-action on $\mathrm{S}_\xi M$ which allows to reinterpret contact geometry as $\mathbb{R}$-equivariant symplectic geometry without any loss of information. On one hand, many contact invariants are constructed from symplectizations using holomorphic curves techniques. It is therefore tempting to think that contact manifolds with symplectomorphic symplectizations are contactomorphic. On the other hand, in smooth topology it is well-known that there exist manifolds M and M' that are not diffeomorphic but for which $\mathbb{R} \times M$ and $\mathbb{R} \times M'$ are diffeomorphic (see [2]). Using flexibility results of Eliashberg and Cieliebak [4], we can realize these examples in a symplectic setting to construct non-diffeomorphic contact manifolds with symplectomorphic symplectizations.

Definition 1.1. Let $(M, \xi = \ker \alpha)$ be a contact manifold. The symplectic manifold $(\mathrm{S}_\xi M, \omega_\xi) = (\mathbb{R} \times M, d(e^t \alpha))$ is called the *symplectization* of (M, ξ). It is endowed with an $\mathbb{R}$-action given by translation in the $\mathbb{R}$ factor.

Proposition 1.2. *Any $\mathbb{R}$-equivariant symplectomorphism $\mathrm{S}_\xi M \to \mathrm{S}_{\xi'} M'$ induces a contactomorphism $(M, \xi) \to (M', \xi')$.*

Now if we relax the hypothesis that the symplectomorphism is $\mathbb{R}$-equivariant in the proposition above, does it still follow that M and M' are contactomorphic?

2. Main results

Theorem 2.1. *[1] For any closed contact manifold (M, ξ) of dimension at least 5 and any closed manifold M' such that $\mathbb{R} \times M$ and $\mathbb{R} \times M'$ are diffeomorphic, there is a contact structure ξ' on M' such that $S_\xi M$ and $S'_\xi M'$ are symplectomorphic.*

Outline of proof. The diffeomorphism $\Psi : \mathbb{R} \times M \to \mathbb{R} \times M'$ produces two h-cobordisms (W, M, M') and (W', M', M) such that the compositions in both senses are trivial :

$$W \cup W' \simeq [0,1] \times M \text{ and } W' \cup W \simeq [0,1] \times M'$$

(for example, W is obtained as the region in $\mathbb{R} \times M$ between $\{0\} \times M$ and $\Psi^{-1}(\{c\} \times M')$ for a sufficiently large positive number c).

Using Eliashberg and Cieliebak's results from [4] we can endow W and W' with *flexible* symplectic structures that induce the contact structure ξ on M and a new contact structure ξ' on M' and we still have, now *symplectically*:

$$W \cup W' \simeq [0,1] \times M \text{ and } W' \cup W \simeq [0,1] \times M'.$$

We apply the Mazur trick (see [2]) and consider the infinite composition V:

$$\cdots (W \cup W') \cup (W \cup W') \cdots = \cdots (W' \cup W) \cup (W' \cup W) \cdots$$

We get from the left hand side that V symplectomorphic to $S_\xi M$ and from the right hand side that V is symplectomorphic to $S_{\xi'} M'$. □

For example, let us consider $M = L(7,1) \times S^2$ endowed with the canonical contact structure ξ coming from the unit tangent bundle of $L(7,1)$. It was proved by Milnor (see [3]) that M is not diffeomorphic to $M' = L(7,2) \times S^2$ but they are h-cobordant. It follows from the s-cobordism theorem and the Mazur trick as in the proof above that $\mathbb{R} \times M$ and $\mathbb{R} \times M'$ are diffeomorphic (see [2]). Hence theorem 2.1 provides a contact structure ξ' on M' such that $\mathrm{S}_\xi M$ and $\mathrm{S}_{\xi'} M'$ are symplectomorphic.

We now discuss an application of this result to the symplectic topology of Stein manifolds. Stein manifolds (of finite type) admit contact manifolds at infinity, given by level sets above any critical value of positive proper plurisubharmonic functions. However we may wonder if this contact manifold depends only on the Stein manifold or may change when we pick a different proper plurisubharmonic function. Again using results from [4] to go from Weinstein to Stein, we can apply the method of Theorem 2.1 to provide different contact boundaries for a given Stein manifold.

Corollary 2.2. *[1] Let V be a Stein manifold of finite type. Let (M, ξ) be the contact manifold at infinity given by a plurisubharmonic function ϕ. Then for any closed manifold M' such that $\mathbb{R} \times M$ and $\mathbb{R} \times M'$ are diffeomorphic, there is a plurisubharmonic function ψ on V with contact manifold at infinity diffeomorphic to M'.*

3. Questions

Does there exist contact structures ξ and ξ' on a given manifold M such that ξ and ξ' are not conjugated by a diffeomorphism of M but $S_\xi M$ and $S'_\xi M$ are symplectomorphic? Any contact invariant which is functorial with respect to symplectic cobordisms (such as contact homology) could not distinguish between ξ and ξ'.

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