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Index theory for basic Dirac operators on Riemannian foliations

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Let $(M, \mathcal{F})$ be a smooth, closed manifold endowed with a Riemannian foliation. Let $D_b^E : \Gamma_b(M, E^+) \to \Gamma_b(M, E^-)$ be a basic, transversally elliptic differential operator acting on the basic sections of a foliated complex vector bundle E over M, of rank N. The basic index $\operatorname{ind}_b\left(D_b^E\right)$ is known to be a well-defined integer, and it has been an open problem since the 1980s to write this integer in terms of the geometric and topological data. Our main theorem expresses $\operatorname{ind}_b\left(D_b^E\right)$ as a sum of integrals over the different strata of the Riemannian foliation, and it involves the eta invariant of associated equivariant elliptic operators on spheres normal to the strata. The result is

$$\operatorname{ind}_b\left(D_b^E\right) = \int_{\widetilde{M_0}\diagup\overline{\mathcal{F}}} A_{0,b}(x)\ \widetilde{|dx|} + \sum_{j=1}^{r} \beta(M_j)\ ,$$

$$\beta(M_j) = \frac{1}{2}\sum_{\tau}\frac{1}{n_\tau \text{rank } W^\tau}\left(-\eta\left(D_j^{S+,\tau}\right) + h\left(D_j^{S+,\tau}\right)\right)\int_{\widetilde{M_j}\diagup\overline{\mathcal{F}}} A_{j,b}^{\tau}(x)\ \widetilde{|dx|}\ .$$

Here, the integrands $A_{0,b}(x)$ and $A_{j,b}^{\tau}(x)$ are the familiar Atiyah-Singer integrands corresponding to local heat kernel supertraces of induced elliptic operators over closed manifolds, $D_j^{S+,r}$ is a first order differential operator on a round sphere, explicitly computable from local information provided by the operator and the foliation, while $\eta(D_j^{S+,r})$ and $h(D_j^{S+,r})$ denote its eta-invariant and kernel, respectively. Even in the case when the operator D is elliptic, such a result was not known previously. We emphasize that every part of the formula is local in the data, even $\eta(D_j^{S+,r})$ is calculated directly from the principal transverse symbol of the operator D_b^E at any point of a singular stratum. The de Rham operator provides an important example illustrating the computability of the formula, yielding the basic Gauss-Bonnet Theorem.

The Theorem is proved by first writing $\operatorname{ind}_b\left(D_b^E\right)$ as the invariant index of a G-equivariant, transversally elliptic operator $\mathcal{D}$ over a G-manifold $\widehat{W}$

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associated to the foliation, where G is a compact Lie group of isometries. Precisely, we lift the given foliation $\mathcal{F}$ to a foliation $\widehat{\mathcal{F}}$ on the principal frame bundle, $\widehat{M}$, associated to $Q \otimes E$, with structure group $G := O(q) \times U(N)$, where Q is the normal bundle to $\mathcal{F}$ of rank q. Then $\mathcal{F}$ is transversally parallelizable. Hence we deduce from Molino's structure theory that the leaf closures of $\widehat{\mathcal{F}}$ are the fibers of the *basic fibration* $\widehat{\pi} : \widehat{M} \to \widehat{W}$. Since G acts isometrically on $\widehat{W}$, we have reduced the problem to the computation of a G-equivariant index. Using our equivariant index theorem, we obtain an expression for this index in terms of the geometry and topology of $\widehat{W}$ and then rewrite this formula in terms of the original data on the foliation.

We note that a recent paper of Gorokhovsky and Lott addresses this transverse index question on Riemannian foliations in a rather special case. Using a different technique, the authors prove a formula for the index of a basic Dirac operator that is distinct from our formula, assuming that all the infinitesimal holonomy groups of the foliation are connected tori and that Molino's commuting sheaf is abelian and has trivial holonomy. Our result requires only mild topological assumptions on the transverse structure of the strata of the Riemannian foliation. In particular, the Gauss-Bonnet Theorem for Riemannian foliations is a corollary and requires no assumptions on the structure of the Riemannian foliation.

We add some remarks on the proof of the Theorem. Using the notions of basic sections, holonomy-equivariant vector bundles, basic Clifford bundles, and basic Dirac-type operators, we describe the Fredholm properties of these basic operators, and we show how to construct the G-manifold $\widehat{W}$ of leaf closures and the G-equivariant operator $\mathcal{D}$, using a slight generalization of Molino theory. We also use our construction to obtain asymptotic expansions and eigenvalue asymptotics of transversally elliptic operators on Riemannian foliations, which are of independent interest. We also construct bundles associated to represention of the isotropy subgroups of the G-action; these bundles are used in the main theorem. In the course of the proof, we describe the construction of the *desingularization* of a Whitney stratified space, i. e. a method of cutting out tubular neighborhoods of the singular strata and doubling the remainder to produce a Whitney stratified space with fewer strata. We also deform the operator and the metric and determine the effect of this desingularization and deformation operation on the basic index. Finally, we prove a generalization of this theorem to representation-valued basic indices.

The theorem is illustrated with a collection of examples. These include foliations by suspension, a Transverse Signature, and the Basic Gauss-Bonnet Theorem.

One known application of our theorem is Kawasaki's Orbifold Index Theorem. It is known that every orbifold is the leaf space of a Riemannian foliation, where the leaves are orbits of an orthogonal group action such that all isotropy subgroups have the same dimension. In particular, the contributions from the eta invariants in our Transverse Signature Theorem agree exactly with the contributions from the singular orbifold strata when the orbifold is four-dimensional.

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