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Riemannian manifolds not quasi-isometric to leaves in codimension one foliations

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The question of when an open (i.e. noncompact) connected manifold can be realized up to diffeomorphism as a leaf in a foliation of a compact differentiable manifold was first posed by Sondow in 1975 for surfaces in 3-manifolds, and was solved positively for all open surfaces by Cantwell and Conlon [3] in 1987. In the opposite direction, in 1985 Ghys [4] and (independently) Inaba, Nishimori, Takamura and Tsuchiya [5] constructed open 3-manifolds—infinite connected sums of lens spaces—that cannot be leaves in a foliation of a compact 4-manifold. Attie and Hurder [2] in 1996 gave an uncountable family of smooth simply connected 6-dimensional manifolds that are not diffeomorphic to leaves in a compact 7-manifold. It is still an open problem whether every smooth open manifold of dimension greater than 2 is diffeomorphic to a leaf of a foliation of codimension two or greater.

For the related question of when an open Riemannian manifold can be realized up to quasi-isometry as a leaf in a foliation of a compact manifold, Attie and Hurder [2] also produced an uncountable family of quasi-isometry types of Riemannian metrics on the 6-manifold $S^3 \times S^2 \times \mathbb{R}$, each with bounded geometry, which are not quasi-isometric to leaves in any codimension one foliation of a compact 7-manifold. (Note that leaves of a foliation on a compact Riemannian manifold must have bounded geometry, such as injectivity radius and curvatures.) Their results extend to codimension one foliations of dimensions greater than 6, but they asked (Question 2 in [2]) whether there exist examples in the lower dimensions $3, 4$, and 5. We answer this question in the following theorem.

Theorem 1. *Every connected non-compact smooth p-manifold L of dimension $p \geq 2$ possesses C^∞ complete Riemannian metrics g with bounded geometry that are not quasi-isometric to any leaf of a codimension one $C^{2,0}$ foliation on any compact differentiable $(p+1)$-manifold.*

Furthermore g can be chosen such that no end is quasi-isometric to an end of a leaf of such a foliation, and also to have any growth type compatible with bounded geometry. Hence there are uncountably many quasi-isometry classes of such metrics g on every such manifold L.

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Consequently no bounded local geometric invariants of an open Riemannian p-manifold with $p \geq 2$ can be obstructions to its being quasi-isometric to a leaf.

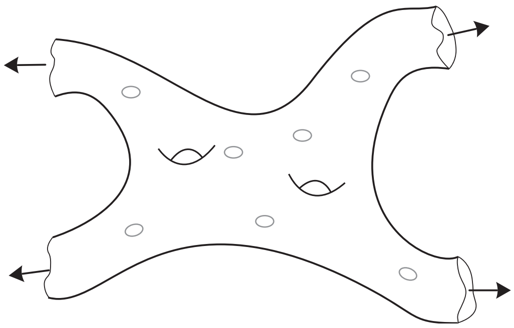

Figure 1: The manifold L with the original metric.

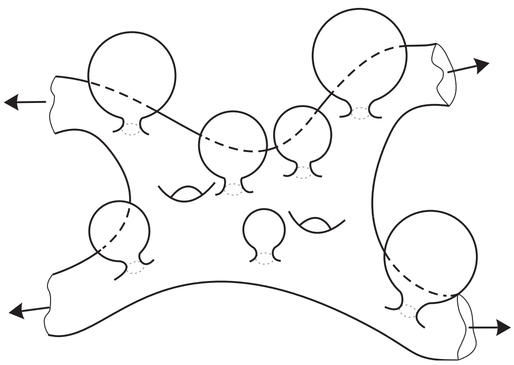

Figure 2: The manifold L with "balloons".

Our construction of Riemannian metrics on open manifolds modifies an arbitrary given metric by replacing open disks of a fixed small radius δ by large balloons, which are the complements of δ-disks in spheres of arbitrarily large radius. This can be done so that the curvature and injectivity radius remain globally bounded. By inserting these balloons in disks that converge rapidly to the ends of the manifold, the original growth rate can be preserved. We show that open manifolds with arbitrarily large balloons cannot be codimension one leaves in a compact manifold.

For surfaces the theorem was proven in [7] using a certain 'bounded homotopy property'. For manifolds of dimension $p \geq 3$, we introduce an analogous 'bounded homology property' which must be satisfied for all leaves of $C^{2,0}$ codimension one foliations of compact $(p+1)$-manifolds [8]. The essential idea is that an embedded vanishing cycle of limited size that bounds on its p-dimensional leaf must bound a region C of the leaf that in a certain sense is 'small'. From consideration of the leaves in a Reeb component it is obvious that the p-volume of the bounded region may be arbitrarily large. Hence we need a different notion of size. We consider non-negative Morse functions on the region C that vanish on its boundary (the vanishing cycle), and we require that there be a Morse function whose level sets have bounded $(p-1)$-volume. The minimum over all such Morse functions of the maximum $(p-1)$-volume of the level sets is called the *Morse volume* of the bounded set C. A manifold L possesses the *bounded homology property* if, for every constant $K > 0$ there is an $\epsilon >$ such that for every connected embedded cycle that bounds on L and has volume less than ϵ, the manifold it bounds must have Morse volume less than K.

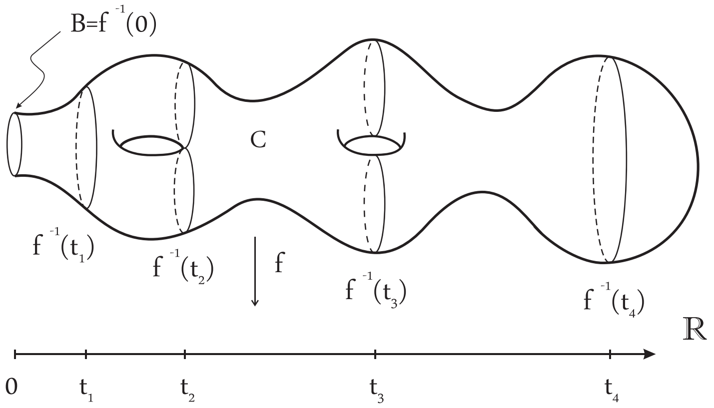

Figure 3: Morse volume of a compact manifold C with boundary B.

It is easy to see that the leaves in a Reeb component have uniformly bounded Morse volume.

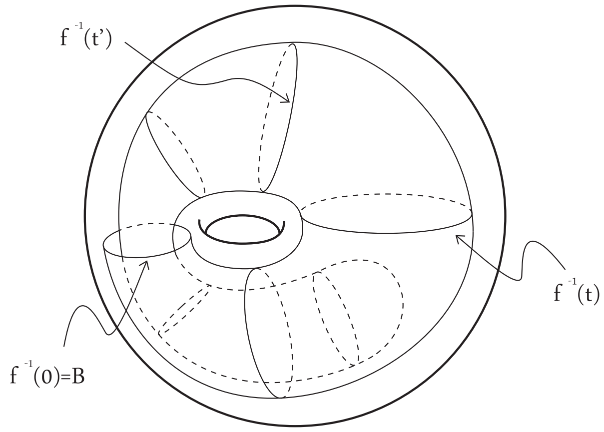

Figure 4: The Morse volume of a set C in a leaf of a Reeb component.

We prove that leaves in a codimension one foliation of a compact $(p+1)$-manifold $(p \geq 3)$ must have the bounded homology property. It is clear that a Riemannian manifold in which arbitrary large balloons have been inserted does not have the bounded homology property, thus showing the theorem. At a certain point in the proof of the theorem we need a weak generalization of Novikov's celebrated theorem on the existence of Reeb components: Every connected $(p-1)$-dimensional vanishing cycle *embedded* on a p-dimensional leaf in a compact foliated $(p+1$-manifold must lie on the boundary of a (generalized) Reeb component with connected boundary; a generalized Reeb component with connected boundary is defined to be a compact foliated $(p+1)$-manifold with connected non-empty boundary whose interior foliates over the circle with the leaves as fibers. We give a proof of this weak generalization.

A diffeomorphism $f : L \to L'$ between two Riemannian manifolds L and L' is defined to be a **quasi-isometry** if there exist constants $C, D > 0$ such that the distance functions d and d' on L and L' satisfy

$$C^{-1}d'(f(x), f(y)) - D \leq d(x, y) \leq Cd'(f(x), f(y)) + D$$

for all points $x, y \in L$. For example, any diffeomorphism between compact smooth Riemannian manifolds is a quasi-isometry. The presence of the constant $D > 0$ in this definition requires some technical details in the definition of the bounded homology property so that it will be an invariant of quasi-isometry.

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