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On the space of left-orderings of solvable groups

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1. Introduction

A left-orderable group is a group Γ which admits a total ordering $\preceq$ invariant under multiplications, that is, $f \prec g \Rightarrow hf \prec hg$ for all $f, g, h \in \Gamma$. Equivalently Γ is left-orderable if we can find $P \subset \Gamma$ satisfying

i) $P\,P \subseteq P$, so P is a semigroup.

ii) $\Gamma = P \sqcup P^{-1} \sqcup \{id\}$, where the unions are disjoint.

The set P is usually called the positive cone of an ordering $\preceq$, since the equivalence between the two above definitions is given by $P_{\preceq} = \{f \in \Gamma \mid f \succ id\}$.

Given a left-orderable group Γ, we shall denote by $\mathcal{LO}(\Gamma)$ its associated space of left-orderings, which consists of all possible left-orderings on Γ. A natural topology can be put in $\mathcal{LO}(\Gamma)$ by considering the inclusion $P \mapsto \chi_P \in \{0,1\}^\Gamma$, where χ_P denotes the characteristic function over P, and the topology on $\{0,1\}^\Gamma$ is the product topology. In this way, we have that two left-orderings are close if they coincide on a large finite set. Moreover, one can check that the inclusion $\mathcal{LO}(\Gamma) \to \{0,1\}^\Gamma$ is closed, hence proving

Theorem 1.1 (Sikora [12])**.** *With the above topology, $\mathcal{LO}(\Gamma)$ is compact and totally disconnected. Moreover, if Γ is countable, then $\mathcal{LO}(\Gamma)$ is metrizable.*

It is interesting to observe that if Γ is countable and $\mathcal{LO}(\Gamma)$ has no isolated left-orderings, then $\mathcal{LO}(\Gamma)$ is homeomorphic to the Cantor set. The problem of relating the topology of $\mathcal{LO}(\Gamma)$ with the algebraic structure of Γ has been of increasing interest since the discovery by Dubrovina and Dubrovin that the space of left-orderings of the braid groups is infinite and yet contains isolated points [2]. Recently, more examples of groups showing these two behaviors have appeared in the literature [1, 4, 5, 8]. Although all this groups contain free subgroups, it is known that non-trivial free products of groups have no isolated left-orderings [10]. In the same spirit, it is a result of Navas [7], that for finitely generated groups with subexponential growth (*e.g.* nilpotent groups), the associated space of left-orderings is either finite or homeomorphic to the Cantor set.

2. Main results

In this talk I will try to convince you of the following result.

Theorem A(Rivas-Tessera [11]): *The space of left-orderings of a countable virtually solvable group is either finite or homeomorphic to a Cantor set.*

There are at least three main ingredients, the first one being the notion of convex subgroup of an ordered group (see for instance [6]).

DEFINITION 2.1. A subset C of a left-ordered group $(\Gamma, \preceq)$ is *convex* if the relation $c_1 \prec f \prec c_2$, for c_1 and c_2 in C, implies that $f \in C$.

For us, the main utility of this notion is the following

Proposition 2.2. *Let $\preceq$ be a left-ordering on Γ and let H be a convex subgroup. Then there is a continuous injection $\mathcal{LO}(H) \to \mathcal{LO}(\Gamma)$, having $\preceq$ in its image. Moreover, if in addition H is normal, then there is a continuous injection $\mathcal{LO}(H) \times \mathcal{LO}(G/H) \to \mathcal{LO}(G)$ having $\preceq$ in its image.*

Therefore, to prove Theorem A, given a left-ordering $\preceq$ it is enough to find subgroup H that is convex for $\preceq$ and such that $\mathcal{LO}(H)$ has no isolated left-orderings, or such that H is normal and $\mathcal{LO}(\Gamma/H)$ has no isolated left-orderings.

The second main ingredient is the following nice characterization of left-orderability (see [3])

Proposition 2.3. *For a countable group Γ, the following assertions are equivalent*

- Γ *is left-orderable.*
- Γ *acts faithfully by order preserving homeomorphisms of the real line.*

This puts at our disposal the strong machinery of group actions on the real line. For instance, of mayor importance for us will be the following theorem.

Theorem 2.4 (Plante [9]). *Every finitely generated nilpotent group of $Homeo_+(\mathbb{R})$, acting without global fixed point, preserves a measure on the real line, which is finite on compact sets and has no atoms (a Radon measure for short).*

Finally, the last main ingredient is the notion of Conradian orderings. Recall that a left-ordering $\preceq$ is called Conradian, if in addition it satisfies that $f \succ id$, $g \succ id \Rightarrow fg^2 \succ id$. What it is so important about Conradian orderings is their nice dynamical counterpart discovered by Navas in [7].

Theorem 2.5 (Navas [7]). *Let $\preceq$ be a Conradian ordering on a group Γ. Then, the action on the real line associated to $\preceq$ is an action without crossings.*

The easiest definition of a crossing is the following picture.

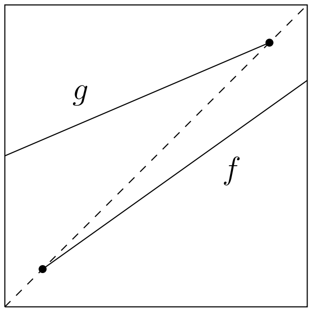

Figure 1: The graphs of the crossed homeomorphisms f and g.

Equivalently, a group $\Gamma \subset Homeo_+(\mathbb{R})$ is said to acts without crossings, if whenever $f \in \Gamma$ fixes a open interval I_f, but has no fixed point in it, then for any $g \in \Gamma$ we have that

$$g(I_f) \cap I_f = \begin{cases} I_f, \text{ or} \\ \emptyset. \end{cases}$$

This three main ingredient will be put to work together in order to show Theorem A. We shall put some emphasis in the case where Γ is a polycyclic group (that is when Γ is finitely generated solvable, and its successive quotient in the derived series are cyclic), which is the simpler non-trivial incarnation of Theorem A.

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