Geometry and Foliations 2013
Komaba, Tokyo, Japan

Rigidity and arithmeticity in Lie foliations

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1. Introduction

Background. For a Lie group G, a *G-Lie foliation* is a foliation transversely modeled on the G-action on G by left translation. Such foliations have been investigated being motivated by the classification of Riemannian foliations (see [6, 8]). The first example of Lie foliations is the following one, which is called *homogeneous*:

Example 1.1. Let G and H be connected Lie groups. Let K be a closed Lie subgroup of H, and Γ a torsion-free cocompact lattice of $H \times G$. Then we have a G-Lie foliation on $K\backslash H \times G/\Gamma$ induced from the product foliation $K\backslash H \times G = \sqcup_{g \in G} K\backslash H \times \{g\}$.

A number of examples of nonhomogeneous Lie foliations were constructed in [15, 16, 9]. On the other hand, under various conditions, minimal Lie foliations tend to be homogeneous or have rigidity which is quite useful for the classification: Caron-Carrière [3] showed that 1-dimensional Lie foliation is diffeomorphic to a linear flow on a torus. Matsumoto-Tsuchiya [14] proved that any 2-dimensional affine Lie foliation on closed 4-manifolds are homogeneous. Zimmer [24] proved that if a minimal G-Lie foliation admits a Riemannian metric such that each leaf is isometric to a product of symmetric space of noncompact type of rank greater than one, then the holonomy group is arithmetic.

Motivation. This work was motivated by the following two questions on rigid aspects of Lie foliations mentioned in the last paragraph.

Question 1.2. Classify minimal $\mathrm{SL}(2;\mathbb{R})$-Lie foliations whose leaves are hyperbolic plane.

By a theorem of Carrière [4], for a G-Lie foliation on a compact manifold, G is solvable if and only if each leaf admits a Følner sequence. Thus Lie foliations in Question 1.2 are of the lowest dimension among Lie foliations with hyperbolic leaves.

[1] Partly supported by by JSPS/IHÉS/EPDI Fellowship

QUESTION 1.3. Classify 3-dimensional minimal Lie foliations which admits leafwise geometrization in the sense of Thurston.

One may conjecture that any 3-dimensional minimal Lie foliations may admits leafwise geometrization. Thus Question 1.3 may be considered as a step for the classification of 3-dimensional minimal Lie foliations.

Main results. This talk is based on work in progress. The main result is the following:

Theorem 1.4. *Let $(M, \mathcal{F})$ be a compact manifold with a minimal G-Lie foliation. Assume that M admits a Riemannian metric such that every leaf of $\mathcal{F}$ is isometric to a symmetric space $X = \prod X_i$, where X_i is an irreducible Riemannian symmetric space of noncompact type of dimension greater than two. Then $(M, \mathcal{F})$ is homogeneous.*

This result gives a complete answer for Question 1.3 in the case where the leaves are $\mathbf{H}^3$. We describe the proof in detail in Section 3. The key step of the proof is to show that the geodesic boundary of hyperbolic leaves admits a $\pi_1 M$-invariant conformal structure thanks to ergodicity of the $\pi_1 M$-action or the leafwise geodesic flow. This phenomenon can be regarded as a certain family version of strong Mostow rigidity for locally symmetric spaces [19]. Our proof is not sufficient to solve Question 1.2 in the same reason why Mostow strong rigidity fails to hold for Riemann surfaces.

We deduce two consequences of Theorem 1.4. We need the following result, which will be proved in Section 4.

Proposition 1.5. *If a homogeneous Lie foliation $(K \backslash H \times G/\Gamma, \mathcal{F})$ in Example 1.1 satisfies the assumption of Theorem 1.4, then G is semisimple and the projection of Γ to any connected normal subgroup of $H \times G$ is dense.*

A lattice Γ of a connected Lie group G is called *superrigid* if, for any real algebraic group H containing no connected simple compact normal subgroups, any homomorphism $\Gamma \to H$ with Zariski dense image virtually extends to a continuous homomorphism $G \to H$. Combining Theorem 1.4, Proposition 1.5 with an extension of Margulis' superrigidity theorem due to Starkov [23, Theorem 4.6], we get the following.

Corollary 1.6. *Under the assumption of Theorem 1.4, $\pi_1 M$ is isomorphic to a superrigid cocompact lattice in $H \times G$.*

Combining with Theorem 1.4 and Proposition 1.5 with Margulis' arithmeticity theorem [12, Theorem A in p. 298], we get the following consequence, which implies a generalization of a theorem of Zimmer [24, Theorem A-3], which says that the holonomy group of Lie foliation whose leaves are isometric to a product of symmetric space of noncompact type of rank greater than one is arithmetic.

Corollary 1.7. *In addition to the assumption of Theorem 1.4, we assume that X is of rank greater than one. Then $\pi_1 M$ is isomorphic to an S-arithmetic subgroup of $H \times G$.*

The advantage of arithmeticity is that arithmetic subgroups can be listed up in a sense. Thus Lie foliations in Corollary 1.7 are classified in a sense.

2. Questions

The following is related to Question 1.2.

Question 2.1. Does there exist a non-homogeneous minimal Lie foliation on a closed manifold whose leaves are isometric to hyperbolic planes?

The following is a question concerning the possibility of generalizations of a theorem of Matsumoto-Tsuchiya [14] on homogeneity of solvable Lie foliations.

Question 2.2. Find a good condition which implies the rigidity of minimal G-Lie foliations when G is solvable.

Tits buildings are arithmetic analog of symmetric spaces which have similar rigidity theoretic properties.

Question 2.3. Construct minimal Lie foliations whose leaves are quasi-isometric to Tits buildings of rank greater than one. Do they have rigidity?

Question 2.4. The leaves of the example [9, Section 6] of a minimal $\mathrm{SL}(2;\mathbb{R})$-Lie foliation are quasi-isometric to a Tits building of rank one. Does it have rigidity?

3. Outline of the proof of Theorem 1.4

Step I. The leafwise boundary of foliations. First we explain the proof of Theorem 1.4 in the case where $X = \mathbf{H}^n_{\mathbb{R}}$ $(n \geq 3)$.

Let $(M, \mathcal{F})$ be a minimal G-Lie foliation on a compact manifold. Assume that M admits a Riemannian metric such that each leaf of $\widetilde{\mathcal{F}}$ is isometric to X. Let $(\widetilde{M}, \widetilde{\mathcal{F}})$ be the universal cover of $(M, \mathcal{F})$. Let $\mathcal{G}(\widetilde{\mathcal{F}}) = \{\gamma \mid \gamma$ is a geodesic in a leaf of $\widetilde{\mathcal{F}}\}$. We define the leafwise geodesic boundary $\partial\widetilde{\mathcal{F}}$ of $(\widetilde{M}, \widetilde{\mathcal{F}})$ by

$$\partial\widetilde{\mathcal{F}} = \mathcal{G}(\widetilde{\mathcal{F}})/\sim,$$

where $\gamma \sim \gamma'$ if and only if γ and γ' are contained in a leaf of $\widetilde{\mathcal{F}}$ and asymptotic to each other. By the structure theory of Lie foliations (see [17, Section 4.2]), we have an X-bundle $\mathrm{dev} : \widetilde{M} \to G$ whose fibers are the leaves of $\widetilde{\mathcal{F}}$. We also have a homomorphism $\mathrm{hol} : \pi_1 M \to G$, which makes dev a $\pi_1 M$-equivariant X-bundle. Then $\partial\widetilde{\mathcal{F}}$ is the total space of a $\pi_1 M$-equivariant ∂X-bundle $\partial\,\mathrm{dev} : \partial\widetilde{\mathcal{F}} \to G$, where ∂X is the geodesic boundary of X.

Step II. Ergodicity of the $\pi_1 M$-action on the leafwise boundary. Let $H = \mathrm{Isom}\, X = \mathrm{PSO}(n, 1)$ and K the isotropy group of a point on X so that $X = K\backslash H$. Since each leaf of $\mathcal{F}$ is isometric to X, we have a canonical K-principal bundle $N \to M$ over M with an isometric H-action. Here we have $\partial\widetilde{\mathcal{F}} = \widetilde{N}/P$ for a parabolic subgroup P. Thus $\partial\widetilde{\mathcal{F}}$ has a Lebesgue measure. The following is the key step in the case where $X = \mathbf{H}^n_{\mathbb{R}}$.

Proposition 3.1. *The $\pi_1 M$-action on $\partial\widetilde{\mathcal{F}}$ constructed in Step I is ergodic with respect to the Lebesgue measure.*

In the sequel, we consider Lebesgue measures on smooth manifolds. We will use the following results.

Lemma 3.2 (A modification of [18, Proposition 4])**.** *Let Γ_1 and Γ_2 be two groups. Let Z be a smooth manifold with a $(\Gamma_1 \times \Gamma_2)$-action such that Z/Γ_1 and Z/Γ_2 are smooth manifolds. Then the Γ_1-action on Z/Γ_2 is ergodic if and only if the Γ_2-action on Z/Γ_1 is ergodic.*

Theorem 3.3 (A part of [18, Theorem 1])**.** *Let H be a semisimple Lie group with no compact connected subgroup. Let P be a subgroup of H. Then the following are equivalent:*

1. *The image of P under the projection from H to each connected simple normal subgroup of H is noncompact.*
2. *For any unitary H-representation π in the Hilbert space V and any vector $v \in V$, if $\pi(P)v = v$, then $\pi(H)v = v$.*

Proof of Proposition 3.1. By Lemma 3.2, the $\pi_1 M$-action on $\partial\widetilde{\mathcal{F}}$ is ergodic if and only if the P-action on N is ergodic. By Theorem 3.3 for

Hilbert space $L^2(N)$, any P-invariant L^2-function on N is H-invariant. Thus the latter condition is equivalent to the ergodicity of the H-action on N. Lemma 3.2 implies that the H-action on N is ergodic if and only if the Γ-action on $\widetilde{N}/H = G$ is ergodic. Since Γ is a dense subgroup of G, the Γ-action on G is ergodic (see the proof of [18, Proposition 4]). $\square$

Step III. Construction of a homomorphism $\pi_1 M \to H$. We construct a trivialization of $\partial\widetilde{\mathcal{F}}$ as a ∂X-bundle over G based on the construction [9, Section 3]. For $g \in G$, denote the leaf of $\widetilde{\mathcal{F}}$ which is the fiber of dev over g by $L(g)$. Take a $\pi_1 M$-invariant metric on $(\widetilde{M}, \widetilde{\mathcal{F}})$. Any left invariant vector field ξ on G can be horizontally lifted to $\widetilde{M}$ along $\mathrm{dev} : \widetilde{M} \to G$ so that the lift $\widetilde{\xi}$ is tangent to $(T\widetilde{\mathcal{F}})^{\perp}$. Since dev is $\pi_1 M$-equivariant and M is compact, the flow on $\widetilde{M}$ generated by $\widetilde{\xi}$ is bi-Lipschitz. For each $g \in G$, take the left invariant vector field ξ on G such that $\exp \xi = g$. By the flow on $\widetilde{M}$ generated by $\widetilde{\xi}$, we have a map $\Phi(g) : \widetilde{M} \to \widetilde{M}$ whose restriction to $L(h)$ is bi-Lipschitz for any $h \in G$. Here $\Phi(g)$ induces a map $\partial\Phi(g) : \partial\widetilde{\mathcal{F}} \to \partial\widetilde{\mathcal{F}}$ whose restriction to $\partial L(h)$ is a quasi-conformal homeomorphism (see [19, Section 21]). Clearly we have $\partial\Phi(g_1) \circ \partial\Phi(g_2) = \partial\Phi(g_1 g_2)$. Then we get a trivialization $\partial\widetilde{\mathcal{F}} \cong \partial L(e) \times G$. Let e_G be the unit element of G. we obtain a $\pi_1 M$-action on $\partial L(e_G)$ given by

$$\begin{array}{ccc} \pi_1 M \times \partial L(e_G) & \longrightarrow & \partial L(e_G) \\ (c, [\gamma]) & \longmapsto & \partial\Phi(\mathrm{hol}(c)^{-1})([c \cdot \gamma]) \, , \end{array} \tag{3.4}$$

where $\cdot$ denotes the $\pi_1 M$-action on the space $\mathcal{G}(\widetilde{\mathcal{F}})$ of geodesics.

Since a quasi-conformal homeomorphism is absolutely continuous, the trivialization preserves the Lebesgue measure class. Thus, by Proposition 3.1, we have ergodicity of (3.4). Here we apply the following.

Proposition 3.5 ([19, Section 22])**.** *Let $n \geq 2$ and $q : S^n \to S^n$ be a quasi-conformal homeomorphism. If q is equivariant with respect to an ergodic group action, then q is conformal.*

Then we conclude that the $\pi_1 M$-action (3.4) on $\partial L(e_G)$ is conformal. We get a homomorphism $\pi_1 M \to \mathrm{Conf}(\partial X) \cong \mathrm{Isom}\, X = H$.

Step IV. Construction of a homogeneous Lie foliation $(M_0, \mathcal{F}_0)$. Let $\rho : \pi_1 M \to H$ be the homomorphism constructed in Step III. Consider the direct product $\rho \times \mathrm{hol} : \pi_1 M \longrightarrow H \times G$. Let $\Gamma = (\rho \times \mathrm{hol})(\pi_1 M)$.

We show that Γ is discrete in $H \times G$. Assume that Γ is not discrete. Then there exists a sequence $\{c_k\}$ in $\pi_1 M$ such that $\rho(c_k) \to e_H$ and $\mathrm{hol}(c_k) \to e_G$. Let $\psi_k : L(e_G) \to L(e_G)$ be the isometry which induces a conformal transformation $\rho(c_k)$ on $\partial L(e_G)$. Take a point x in $L(e_G)$ and consider a sequence $\{a_k\}$ in $L(e_G)$ defined by

$$a_k = \psi_k^{-1}\big(\Phi(\mathrm{hol}(c_k)^{-1})(c_k \cdot x)\big) ,$$

where $\Phi(\mathrm{hol}(c_k)^{-1}) : L(\mathrm{hol}(c_k)) \to L(e_G)$ is the bi-Lipschitz map constructed in Step III and $\cdot$ denotes the $\pi_1 M$-action on $\widetilde{M}$. By construction, the map

$$\begin{array}{rccc} \chi_k : & L(e_G) & \longrightarrow & L(e_G) \\ & y & \longmapsto & \psi_k^{-1}\big(\Phi(\mathrm{hol}(c_k)^{-1})(c_k \cdot y)\big) \end{array}$$

is a bi-Lipschitz map which induces the identity on $\partial L(e_G)$. Since $\{\mathrm{hol}(c_k)\}$ converges to e_G, there exists a positive number C such that, for any k, $\Phi(\mathrm{hol}(c_k)^{-1})|_{L(\mathrm{hol}(c_k))}$ is bi-Lipschitz with Lipschitz constant C. Then, χ_k is a bi-Lipschitz with Lipschitz constant C for any k. By the Morse lemma (see, for example, [2, 8.4.20]), there exists $r > 0$ such that χ_k maps any geodesic τ in $L(e_G)$ into an r-neighborhood of τ. This implies that $d(y, \chi_k(y)) < r$, where d is the distance on $L(e_G)$. Then $\{\chi_k(x)\}$ admits a converging subsequence. By construction, this implies that $\{c_k \cdot x\}$ admits a converging subsequence. This contradicts with the properly discontinuity of the $\pi_1 M$-action on $\widetilde{M}$. Thus Γ is discrete in $H \times G$.

We show that Γ is cocompact in $H \times G$. We denote the real cohomological dimension of manifolds and groups by rcd. First we compute $\mathrm{rcd}\,\Gamma$. By applying [5, Lemme 2.4] to $\mathrm{dev} : \widetilde{M} \to G$, we have

$$\mathrm{rcd}\,\widetilde{M} \leq \mathrm{rcd}\,\widetilde{L} + \mathrm{rcd}\,G ,$$

$$\mathrm{rcd}\,M \leq \mathrm{rcd}\,\widetilde{M} + \mathrm{rcd}\,\Gamma ,$$

where $\widetilde{L}$ is a leaf of $\widetilde{\mathcal{F}}$. Since $\widetilde{L}$ is contractible, $\mathrm{rcd}\,\widetilde{L}$ is zero. Since M is compact, we have $\mathrm{rcd}\,M = \dim M$. Thus we get

$$\mathrm{rcd}\,G + \mathrm{rcd}\,\Gamma \geq \dim M .$$

Let K_G be a maximal compact subgroup of G. Let $X_G = K_G \backslash G$. Recall that K is a maximal compact subgroup of H such that $X = K \backslash H$. Since $\mathrm{rcd}\,G = \dim K$, $\dim G = \dim X_G + \dim K$ and $\dim M = \dim G + \dim X$. We get

$$\mathrm{rcd}\,\Gamma \geq \dim X_G + \dim X .$$

On the other hand, since a finite index subgroup of Γ acts freely on $X \times X_G$ which is contractible, we get

$$\mathrm{rcd}\,\Gamma \leq \dim X_G + \dim X .$$

Thus we get $\mathrm{rcd}(\Gamma) = \dim X_G + \dim X$. This implies that $H^n\big((X \times X_G)/\Gamma; \mathbb{R}\big)$ is nontrivial, where $n = \dim(X \times X_G)/\Gamma$. Thus Γ is cocompact in $H \times G$.

Then $M_0 = K\backslash H \times G/\Gamma$ is a closed manifold. Here M_0 admits a G-Lie foliation $\mathcal{F}_0$ which is induced from the product foliation $K\backslash H \times G = \sqcup_{g\in G} K\backslash H \times \{g\}$ and whose leaves are isometric to $X = K\backslash H$.

Step V. Construction of a diffeomorphism. Here we will show that $(M, \mathcal{F})$ is diffeomorphic to $(M_0, \mathcal{F}_0)$. Since $(M, \mathcal{F})$ and $(M_0, \mathcal{F}_0)$ are classifying spaces of G-Lie foliations with the same holonomy group as explained in the last paragraph, there exist smooth maps $f : M \to M_0$ and $f_0 : M_0 \to M$ such that $f^*\mathcal{F}_0 = \mathcal{F}$, $f_0^*\mathcal{F} = \mathcal{F}_0$, $f_0 \circ f \simeq \mathrm{id}_M$ and $f \circ f_0 \simeq \mathrm{id}_{M_0}$. Let $\widetilde{f}$ and $\widetilde{f_0}$ be lifts of f and f_0 to the universal covers. Since M and M_0 are compact, by using $\widetilde{f_0}$, we can show that $\widetilde{f}$ is a quasi-isometry on each leaf. Thus $\widetilde{f}$ induces a $\pi_1 M$-equivariant homeomorphism $\partial\widetilde{f} : \partial\widetilde{\mathcal{F}} \to \partial\widetilde{\mathcal{F}_0}$ which is quasi-conformal on the geodesic boundary of each leaf. The $\pi_1 M$-equivalence of $\partial\widetilde{f}$ and Proposition 3.5 imply that $\partial\widetilde{f}$ is conformal on the geodesic boundary of each leaf. Since $H = \mathrm{Isom}\, X$, for each $g \in G$, there is a unique way to extend $\partial\widetilde{f}|_{\partial L(g)}$ to an isometry on L_g. It is easy to see that, by this extension, we get a well-defined $\pi_1 M$-equivariant diffeomorphism $\widetilde{f_1} : \widetilde{M} \to \widetilde{M_0}$. Thus the proof is concluded.

The case where X is an irreducible symmetric space of rank one. Now X is one of the following: $\mathbf{H}^n_{\mathbb{R}}$, $\mathbf{H}^n_{\mathbb{C}}$, $\mathbf{H}^n_{\mathbb{H}}$ and $\mathbf{H}^2_{\mathbb{O}}$. In the case where $X = \mathbf{H}^n_{\mathbb{C}}$ ($n \geq 2$), Theorem 1.4 is proved in a way similar to the real hyperbolic case by replacing $\mathbf{H}^n_{\mathbb{R}}$ with $\mathbf{H}^n_{\mathbb{C}}$ and by using quasi-conformal mappings over $\mathbb{C}$ (see [19, Section 21]).

If $X = \mathbf{H}^n_{\mathbb{H}}$ or $\mathbf{H}^2_{\mathbb{O}}$, then Theorem 1.4 is proved in a way simpler than the above two cases thanks to the following result of Pansu.

Theorem 3.6 ([20]). *For any quasi-isometry φ on $\mathbf{H}^n_{\mathbb{H}}$ or $\mathbf{H}^2_{\mathbb{O}}$, there exists an isometry φ_1 such that $\varphi \circ \varphi_1^{-1}$ is bounded.*

By this theorem, we can skip Step II. In Step III, we get a homomorphism $\pi_1 M \to H$ without Step II. In the last step, we do not need to show that ∂f is conformal. The rest of the proof is the same.

The case where X is an irreducible symmetric space of rank $r \geq 2$. We refer to [19] for facts used in this paragraph. A *flat* in X is a totally geodesic flat submanifold of dimension r. Let ∂X be the Furstenberg maximal boundary of X, which is defined as a set of asymptotic classes of flats in X. Here ∂X has a structure of a spherical Tits building whose automorphism group $\mathrm{Aut}(\partial X)$ is isomorphic to H. Theorem 1.4 can be proved in this case by replacing the geodesic boundary of hyperbolic spaces to Tits building ∂X. The following is well known.

Proposition 3.7 (see [19, Section 15]). *Any quasi-isometry on X induces an automorphism of Tits building ∂X.*

We define the leafwise boundary $\partial\widetilde{F}$ like in Step I but by replacing geodesics with flats. We skip Step II. By Proposition 3.7, we get a homomorphism $\pi_1 M \to H$ in Step III without Step II. To show the discreteness Γ in $H \times G$ in Step IV, we need to use the following result instead of Morse lemma:

Theorem 3.8 (A special case of [11, Theorem 1.1.3]). *Let Z be an irreducible symmetric space of noncompact type of rank greater than one. Then, for any bi-Lipschitz self-map with Lipschitz constant C on Z, there exists a homothety on Z at distance less than S, where S is a function of C.*

In the last step, we do not need to show that ∂f is conformal. The rest of the proof of Theorem 1.4 is the same as the case where $X = \mathbf{H}^n_{\mathbb{R}}$.

The general case. By a theorem of Kapovich-Kleiner-Leeb [10], for a quasi-isometry ϕ on $\prod_{i=1}^{\ell} X_i$, there exists a quasi-isometry ϕ_i for each i such that $p_i \circ \phi$ is equal to $\phi \circ p_i$ up to a bounded error. If X_i is $\mathbf{H}^n_{\mathbb{H}}$, $\mathbf{H}^2_{\mathbb{O}}$ or an irreducible symmetric space of rank greater than one for any i, then we finish the proof by applying the above argument to each component.

Assume that $X_i = \mathbf{H}^n_{\mathbb{R}}$ $(n \geq 3)$ or $\mathbf{H}^n_{\mathbb{C}}$ $(n \geq 2)$ for some i. Then, in Step II, we need to show that the $\pi_1 M$-action on the geodesic boundary ∂X_i is ergodic. We consider a subfoliation $\mathcal{F}_i$ of $\mathcal{F}$ which is defined by the X_i-factor in each leaf of $\mathcal{F}$. Since the X_i-factor is determined by the holonomy of the given smooth metric, $\mathcal{F}_i$ is a smooth foliation. Let $H_i = \mathrm{X}_i$ and take a subgroup K_i so that $X_i = H_i/K_i$. Since each leaf of $\mathcal{F}_i$ is isometric to X_i, we have a canonical principal K_i-bundle $W_i \to M$. We can lift the foliation $\mathcal{F}_i'$ horizontally to get an $(H' \times G)$-Lie foliation on W_i, where $H' = H/H_i$. By the structure theorem of Lie foliations [17, Theorem 4.2], the closure of a leaf is a submanifold M_i of M. Here $(M_i, \mathcal{F}_i'|_{M_i})$ is a minimal Lie foliation whose leaves are isometric to X_i. We apply the above Step II for $(M_i, \mathcal{F}_i'|_{M_i})$ to show the ergodicity of the $\pi_1 M_i$-action on the geodesic boundary of a leaf of $\mathcal{F}_i$. This implies that the $\pi_1 M$-action on the geodesic boundary of a leaf of $\mathcal{F}_i$ is ergodic. Applying this argument for each i such that X_i is of rank one, we can get a homomorphism $\pi_1 M \to \mathrm{Isom} \prod X_i = H$ in Step III. The rest of the proof is the same.

4. Proof of Proposition 1.5.

Let $G = L \ltimes R$ be the Levi decomposition of G, where L is semisimple with trivial center and R is solvable and normal in G. By the assumption that the leaves of $\mathcal{F}$ are simply-connected, the H-action on $H \times G/\Gamma$ is free.

Then, by [24, Lemma 5.2], $\Gamma \cap R$ is discrete. Since $R \cap \Gamma$ is discrete and normal in $H \times G$, $R \cap \Gamma$ is central in $H \times G$. Thus, by taking quotient of G and Γ by $R \cap \Gamma$, the proof of Proposition 1.5 can be reduced to the case where $R \cap \Gamma$ is trivial. Let $L = SK$ be the decomposition of L such that $\mathrm{Lie}(S)$ is the sum of noncompact semisimple Lie algebras and $\mathrm{Lie}(K)$ is the sum of compact semisimple Lie algebras. Since Γ is a cocompact lattice of $H \times G$, by a consequence of Auslander's theorem [22, Theorem E.10], $R \cap \Gamma$ is a cocompact lattice of KR. Thus R is compact, hence the identity component R_0 is abelian.

We will show that the projection of Γ to any connected normal simple subgroup of $H \times G$ is dense. Let $p : H \times G \to H \times G/KR$ be the projection. Since KR is compact, $p(\Gamma)$ is a lattice of $H \times G/KR$. Then, since $H \times G/KR$ is a semisimple group without connected compact subgroup, by a well known result (see [21, Theorem 5.22]), $p(\Gamma)$ has a finite index subgroup T such that $T = \prod_{i=1}^{m} T_i$, where T_i is an irreducible lattice of a product of some connected normal simple subgroup of $H \times G/KR$. Since the leaves of $\mathcal{F}$ is simply-connected, the restriction of the projection $H \times G \to G$ to Γ is injective. Hence we get $m = 1$ and S is an irreducible lattice of $H \times G/KR$, which implies that so is $p(\Gamma)$ (see [21, Corollary 5.21]). Then the projection of Γ to any normal simple subgroup of $H \times G$ is dense.

To show that G is semisimple, it suffices to show that R is finite. Since R_0 is abelian, the kernel of $R \to G/[G, G]$ is finite. On the other hand, since Γ is a lattice of $H \times G$ and the projection of Γ to any normal simple subgroup of $H \times G$ is dense, a vanishing theorem of Starkov [23] implies that $\Gamma/[\Gamma, \Gamma] = 0$. Since Γ is dense in G, it follows that $G/[G, G] = 0$. Hence R is finite.

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