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Problems on characteristic classes of foliations

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1. Introduction

The theory of characteristic classes of foliations was initiated by discovery of the Godbillon-Vey class of codimension 1 foliations [14] and a ground-breaking work of Thurston [35] proving that it can vary continuously. Soon after this, Bott and Haefliger [5], and also Bernstein and Rozenfeld [3] presented a general framework for this theory and during the 1970's, it has been developed extensively by many people including Heitsch [17] and Hurder [19]. There also appeared closely related theory of Gelfand and Fuks [11] and that of Chern and Simons [8]. The notions of Γ-structures and their classifying spaces due to Haefliger [18] played a crucial role in this theory and Mather [26] and Thurston [36] obtained many fundamental results by using them.

However there remain many important problems to be solved in future. In this talk, we would like to focus on the following two major problems both of which turn out to be extremely difficult. One is the determination of the homotopy type of the classifying space $\mathrm{B}\Gamma_1$ of Γ_1-structures in the C^∞-category. The other is development of characteristic classes of *transversely symplectic* foliations.

2. Homotopy type of $\mathrm{B}\Gamma_1$

The following is one of the major open problems in foliation theory.

Problem 2.1. Determine the homotopy type of $\mathrm{B}\Gamma_1$. More precisely, determine whether the classifying map

$$\mathrm{GV} : \mathrm{B}\overline{\Gamma}_1 \to K(\mathbb{R}, 3)$$

induced by the Godbillon-Vey class, is a homotopy equivalence or not. Here $\mathrm{B}\overline{\Gamma}_1$ denotes the homotopy fiber of the natural map $w_1 : \mathrm{B}\Gamma_1 \to \mathrm{BGL}(1, \mathbb{R}) = K(\mathbb{Z}/2, 1)$.

In [28], we introduced the concept of *discontinuous invariants* of foliations. One possible approach to the above problem would be the following.

Problem 2.2. Determine whether the homomorphism

$$\mathrm{GV}_k : H_{3k}(\mathrm{B}\overline{\Gamma}_1, \mathbb{Z}) \to \wedge^k_{\mathbb{Z}} \mathbb{R} \ (\cong H_k(\mathbb{R}^\delta; \mathbb{Z}))$$

induced by the discontinuous invariants associated with the Godbillon-Vey class is non-trivial or not.

Let $\mathrm{B}\overline{\Gamma}_1^\omega$ denote the classifying space of transversely oriented *real analytic* Γ_1-structures. Haefliger [18] proved that $\mathrm{B}\overline{\Gamma}_1^\omega$ is a $K(\pi, 1)$ space for certain *perfect* group π.

Problem 2.3. Determine whether the natural map

$$(\mathrm{B}\overline{\Gamma}_1^\omega)^+ \to \mathrm{B}\overline{\Gamma}_1$$

is a homotopy equivalence or not, where $^+$ denotes Quillen's plus construction.

Recall here that Thurston constructed a family of real analytic codimension 1 foliations on a certain 3-manifold by making use of the group

$$\mathrm{SL}(2, \mathbb{R}) *_{\mathrm{SO}(2)} \widetilde{\mathrm{SL}}(2, \mathbb{R})_n \subset \mathrm{Diff}^\omega_+ S^1$$

thereby proving that the homomorphism

$$\mathrm{GV} : H_3(\mathrm{B}\overline{\Gamma}_1^\omega; \mathbb{Z}) \to \mathbb{R}$$

is *surjective*. Here $\widetilde{\mathrm{SL}}(2, \mathbb{R})_n$ denotes the n-fold covering group of $\mathrm{SL}(2, \mathbb{R})$.

In the case of piecewise linear (PL for short) category, Greenberg [15] showed that there is a weak homotopy equivalence

$$\mathrm{B}\overline{\Gamma}_1^{\mathrm{PL}} \sim \mathrm{B}\mathbb{R}^\delta * \mathrm{B}\mathbb{R}^\delta$$

where the right hand side represents the join of two copies of $\mathrm{B}\mathbb{R}^\delta$. It follows that $\mathrm{B}\overline{\Gamma}_1^{\mathrm{PL}}$ is 2-connected and he described the integral homology group of $\mathrm{B}\overline{\Gamma}_1^{\mathrm{PL}}$ completely. It also follows that the higher homotopy groups of this space is highly non-trivial.

By making use of this result, Tsuboi [33] showed that all the discontinuous invariants of $\mathrm{B}\overline{\Gamma}_1^{\mathrm{PL}}$ associated with the discrete Godbillon-Vey class $\in H^3(\mathrm{B}\overline{\Gamma}_1^{\mathrm{PL}}, \mathbb{R})$, defined by Ghys and Sergiescu [13], vanishes.

On the other hand, in a certain case of low differentiability (Lipschitz with bounded variation of derivatives), Tsuboi [34] proved that the second discontinuous invariant

$$\mathrm{GV}_2 : H_6(\mathrm{B}\overline{\Gamma}_1^{\mathrm{Lip,bdd}}, \mathbb{Z}) \to \wedge^2_{\mathbb{Z}} \mathbb{R}$$

is highly non-trivial (in fact its cockernel is a torsion group) where GV is the one he extended to this case.

The Godbillon-Vey class can be defined for transversely holomorphic foliations with trivialized normal bundles and Bott [4] proved that the homomorphism

$$\mathrm{GV}^{\mathbb{C}} : \pi_3(\mathrm{B}\overline{\Gamma}_1^{\mathbb{C}}) \to \mathbb{C}$$

is surjective.

Problem 2.4. Determine the homotopy type of $\mathrm{B}\Gamma_1^{\mathbb{C}}$. More precisely, determine whether the classifying map

$$\mathrm{GV}^{\mathbb{C}} : \mathrm{B}\overline{\Gamma}_1^{\mathbb{C}} \to K(\mathbb{C}, 3)$$

induced by the complex Godbillon-Vey class, is a homotopy equivalence or not.

We refer to a book [1] by Asuke for a recent study of $\mathrm{GV}^{\mathbb{C}}$.

Finally we recall a closely related problem. Let $\mathcal{M}^h(3)$ denote the set of orientation preserving diffeomorphism classes of closed oriented *hyperbolic* 3-manifolds. For any such manifold M, we have its volume $\mathrm{vol}(M)$ and the η-invariant $\eta(M)$ of Atiyah-Patodi-Singer [2]. The combination $\eta + i\,\mathrm{vol}$ gives rise to a mapping

$$\eta + i\,\mathrm{vol} : \mathcal{M}^h(3) \to \mathbb{C}.$$

Problem 2.5 (Thurston ([37], Questions 22, 23). Study the above map. In particular, determine whether the dimension over $\mathbb{Q}$ of the $\mathbb{Q}$-subspace of $i\mathbb{R}$ generated by the second component of the image of the above map is infinite or not.

Recall that any such M defines a homology class $[M] \in H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z})$ and we have the following closely related problem.

Problem 2.6. Determine the image of the map

$$\mathcal{M}^h(3) \to H_3(\mathrm{PSL}(2,\mathbb{C})^\delta; \mathbb{Z}) \overset{(\mathrm{CS}, i\mathrm{vol})}{\longrightarrow} \mathbb{C}/\mathbb{Z}.$$

Problem 2.7. Study the discontinuous invariants of the group $\mathrm{PSL}(2,\mathbb{C})^\delta$ associated with the above classes. In particular, determine the value of the *total Chern Simons invariant* introduced in Dupont [9].

3. Characteristic classes of transversely symplectic foliations

One surprising feature of the Gelfand-Fuks cohomology theory was that

$$\dim H_c^*(\mathfrak{a}_n) < \infty$$

where $\mathfrak{a}_n$ denotes the Lie algebra consisting of all the formal vector fields on $\mathbb{R}^n$. The associated characteristic homomorphism

$$\Phi : H_c^*(\mathfrak{a}_n) \to H^*(\mathrm{B}\overline{\Gamma}_n; \mathbb{R})$$

is now very well understood. In contrast with this, the case of all the *volume preserving* formal vector fields $\mathfrak{v}_n \subset \mathfrak{a}_n$ and that of all the *Hamiltonian* formal vector fields $\mathfrak{ham}_{2n} \subset \mathfrak{a}_{2n}$ are both far from being understood.

Problem 3.1. Compute

$$H_c^*(\mathfrak{v}_n), \quad H_c^*(\mathfrak{v}_n, \mathrm{O}(n)), \quad H_c^*(\mathfrak{ham}_{2n}), \quad H_c^*(\mathfrak{ham}_{2n}, \mathrm{U}(n)).$$

In particular, prove (or disprove) that

$$\dim H_c^*(\mathfrak{v}_n) = \infty, \quad \dim H_c^*(\mathfrak{ham}_{2n}) = \infty.$$

Recall here that there are very few known results concerning this problem. First, Gelfand, Kalinin and Fuks [12] found an *exotic* class

$$\text{GKF class} \in H_c^7(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_8$$

and later Metoki [27] found another exotic class

$$\text{Metoki class} \in H_c^9(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R}))_{14}.$$

On the other hand, Perchik [32] obtained a formula for the Euler characteristic and computed it up to certain degree. It suggests strongly that the cohomology would be infinite dimensional.

Let $\mathrm{B}\Gamma_{2n}^{\mathrm{symp}}$ denote the Haefliger classifying space of transversely symplectic foliations of codimension $2n$.

Problem 3.2. Prove that, under the homomorphism

$$\Phi : H_c^*(\mathfrak{ham}_2, \mathrm{Sp}(2, \mathbb{R})) \to H^*(\mathrm{B}\Gamma_2^{\mathrm{symp}}; \mathbb{R})$$

the GKF class and the Metoki class survive as non-trivial characteristic classes.

Kontsevich [22] introduced a new viewpoint in this situation. He considered two Lie subalgebras

$$\mathfrak{ham}_{2g}^1 \subset \mathfrak{ham}_{2g}^0 \subset \mathfrak{ham}_{2g}$$

consisting of Hamiltonian formal vector fields without constant terms and without constant as well as linear terms, respectively. Then he constructed a homomorphism

$$\Phi : H_c^*(\mathfrak{ham}_{2g}^0, \mathrm{Sp}(2g,\mathbb{R})) \cong H_c^*(\mathfrak{ham}_{2g}^1)^{\mathrm{Sp}} \to H_{\mathcal{F}}^*(M)$$

for any transversely symplectic foliation $\mathcal{F}$ on a smooth manifold M of codimension $2n$, where $H_{\mathcal{F}}^*(M)$ denotes the foliated cohomology group. By using this viewpoint, in a joint work with Kotschick [24] we decomposed the Gelfand-Kalinin-Fuks class as a product

$$\text{GKF class} = \eta \wedge \omega$$

where $\eta \in H_c^5(\mathfrak{ham}_2^0, \mathrm{Sp}(2,\mathbb{R}))_{10}$ is a certain leaf cohomology class and ω denotes the transverse symplectic form.

Conjecture 3.3 (Kotschick-M. [24])**.** *The Metoki class can also be decomposed as a product $\eta' \wedge \omega$ for a certain class $\eta' \in H_c^7(\mathfrak{ham}_2^0, \mathrm{Sp}(2,\mathbb{R}))_{16}$.*

On the other hand, $\mathfrak{ham}_{2g}^0, \mathfrak{ham}_{2g}^1$ can be described as

$$\mathfrak{ham}_{2n}^0 = \widehat{\mathfrak{c}}_n \otimes \mathbb{R}, \quad \mathfrak{ham}_{2n}^1 = \widehat{\mathfrak{c}}_n^+ \otimes \mathbb{R}$$

where $\mathfrak{c}_n$ denotes one of the three Lie algebras (commutative one) in Kontsevich's theory [20][21] of graph homology and $\widehat{\mathfrak{c}}_n$ denotes its completion. Thus the above homomorphim Φ can be written as

$$\Phi : H_c^*(\widehat{\mathfrak{c}}_n^+)^{\mathrm{Sp}} \otimes \mathbb{R} \cong H_c^*(\mathfrak{ham}_{2n}^1)^{\mathrm{Sp}} \to H_{\mathcal{F}}^*(M).$$

Besides the theory of transversely symplectic foliations as above, the graph homology of $\mathfrak{c}_n$ has another deep connection with the theory of *finite type* invariants for homology 3-spheres initiated by Ohtsuki [31] which we briefly recall. Let $\mathcal{A}(\phi)$ denote the commutative algebra generated by vertex oriented connected trivalent graphs modulo the (AS) relation together with the (IHX) relation. This algebra plays a fundamental role in this theory. In fact, the completion $\widehat{\mathcal{A}}(\phi)$ of $\mathcal{A}(\phi)$ with respect to its gradings is the target of the LMO invariant [25].

By using a result of Garoufalidis and Nakamura [10], in a joint work with Sakasai and Suzuki [30] we constructed an injection

$$\mathcal{A}(\phi) \to H_*(\mathfrak{c}_\infty^+)^{\mathrm{Sp}}$$

and defined the "complementary" algebra $\mathcal{E}$ so as to obtain an isomorphism

$$H_*(\mathfrak{c}_\infty^+)^{\mathrm{Sp}} \cong \mathcal{A}(\phi) \otimes \mathcal{E}$$

of bigraded algebras. $\mathcal{E}$ can be interpreted as the dual of the space of all the *exotic* stable leaf cohomology classes for transversely symplectic foliations.

Problem 3.4 (cf. Sakasai-Suzuki-M. [30]). Study the structure of $\mathcal{E}$.

4. Homology of $\mathrm{Diff}^\delta M$ and $\mathrm{Symp}^\delta(M,\omega)$

In general, homology group of the diffeomorphism group $\mathrm{Diff}^\delta M$ of a closed C^∞ manifold M, considered as a discrete group, or that of the symplectomorphism group $\mathrm{Symp}^\delta(M,\omega)$ of a closed symplectic manifold (M,ω), again with the discrete topology, is a widely open research area. One can also consider the real analytic case. Here we present a few problems in the cases of the circle S^1 and closed surfaces.

It was proved in [29] that the natural homomorphism

$$\Phi : H_c^*(\mathcal{X}(S^1), \mathrm{SO}(2))) \cong \mathbb{R}[\alpha,\chi]/(\alpha\chi) \to H^*(\mathrm{BDiff}_+^\delta S^1;\mathbb{R})$$

from the Gelfand-Fuks cohomology of S^1, relative to $\mathrm{SO}(2) \subset \mathrm{Diff}_+ S^1$, to the cohomology of $\mathrm{Diff}_+^\delta S^1$, is injective. Also there were given certain non-triviality results for the associated discontinuous invariants.

Problem 4.1. Prove (or disprove) that the homomorphism

$$\Phi : H_c^*(\mathcal{X}(S^1), \mathrm{SO}(2)) \cong \mathbb{R}[\alpha,\chi]/(\alpha\chi) \to H^*(\mathrm{BDiff}_+^{\omega,\delta} S^1;\mathbb{R})$$

is injective, where $\mathrm{Diff}_+^{\omega,\delta} S^1$ denotes the real analytic diffeomorphism group of S^1 equipped with the discrete topology.

Problem 4.2. Determine whether the natural inclusion

$$\mathrm{Diff}_+^{\omega,\delta} S^1 \to \mathrm{Diff}_+^\delta S^1$$

induces an isomorphism in homology or not.

Of course one can consider the above problem for any closed manifold M.

Let Σ_g denote a closed oriented surface of genus g. Harer stability theorem [16] states that the homology group $H_k(\mathrm{BDiff}_+\Sigma_g)$ is independent of g in a certain stable range $k \ll g$ (see a survey paper [38] by Wahl for more details).

By applying a general method, we can define certain characteristic classes for foliated Σ_g-bundles. Also, in [23] certain characteristic classes for foliated Σ_g-bundles with area-preserving holonomy were defined by making use of the notion of the flux homomorphism. These classes are all *stable* with respect to the genus g and it seems reasonable to present the following.

Problem 4.3. Determine whether certain analogue of Harer stability theorem holds for the group $\mathrm{Diff}^\delta\Sigma_g$ and/or $\mathrm{Symp}^\delta\Sigma_g$.

Bowden [6][7] obtained some interesting results related to this problem.

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