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The normal h-principle for foliations and Mather-Thurston homology equivalence

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This talk introduces a *normal* h-principle for foliations, which is a refinement of Thurston's h-principle. "Normal" means that we prescribe a plane field to which the foliation to be built will be normal, except in some parts of the ambiant manifold, the so-called holes. The interesting case is when this plane field is tangential to the fibres of a fibration. We get:

Theorem 0.1. *Let $f : M^{p+q} \to B^p$ be a fibration between closed manifolds, $q \geq 2$, and let ξ be a Γ_q-structure on M , whose normal bundle is isomorphic to* $\ker Df$. *Then, there is a foliation $\mathcal{F}$ of codimension q on M s.t.*

- *$\mathcal{F}$ is homotopic to ξ as a Γ_q-structure;*
- *$\tau\mathcal{F}$ is a limit of p-plane fields transverse to the fibres.*

More precisely, $\mathcal{F}$ is transverse to the fibres, except along a submanifold of codimension 1 , union of compact leaves bounding some kind of vertical Reeb components, given by Thurston's method to fill holes in codimension larger than 1 . The theorem also holds true when $q = 1$, $p = 2$, and when f is a Seifert fibration.

The same method also gives a new proof of the Mather-Thurston homology equivalence in all codimensions.

The h-principle for foliations is a powerful tool to build foliations on a given manifold M , and to classify them up to concordance. Due to Haefliger for M open, and to Thurston for M compact, it says that every *formal foliation* on M is homotopic to some genuine foliation.

We are interested in the case where M is closed. Recall that a Γ_q-structure on M is a pair $\xi = (\nu\xi, \mathcal{X})$ where

- $\nu\xi$ is a real linear bundle of rank q over M , the *normal bundle,* or *microbundle*;
- $\mathcal{X}$ is a foliation of codimension q on the total space $\nu\xi$, transverse to the fibres (in fact, the germ of such a foliation along the null section).

A formal foliation is a pair (ξ, j) , where ξ is a Γ_q-structure on M , and where $j : \nu\xi \to \tau M$ is a linear bundle monomorphism.

Write $\bar{M} := M \times [0,1]$ and $\bar{M}_i := M \times i$ $(i = 0, 1)$. A *homotopy* between two formal foliations (ξ_i, j_i) on M , sharing the same normal bundle, is a Γ_q-structure ξ on $\bar{M}$ s.t. $\xi | \bar{M}_i = \xi_i$ $(i = 0, 1)$, together with a continuous homotopy of linear bundle monomorphisms $j_t : \nu\xi_0 \to \tau M$ $(t \in [0,1])$.

Recall the sketch of Thurston's proof (skipping technicallities about "good position" and "civilization"). We start from a closed manifold M on which are given a Γ_q-structure ξ and a linear bundle monomorphism $j : \nu\xi \to \tau M$ from the microbundle of ξ into τM . One easily translates these data into an embedding of $\bar{M}$ into a large-dimensional open manifold E and a plane field F of codimension q on E s.t.

- On some neighborhood of $\bar{M}_0$, the field F is integrable, and this foliation induces ξ by restriction to $\bar{M}_0$;
- Along $\bar{M}_1$, the field F is transverse to $j(\nu\xi)$.

Then, $\bar{M}$ is finely triangulated, and jiggled in E , giving a PL submanifold $\bar{M}' \subset E$, C^0-close to $\bar{M}$, s.t. the field F in transverse to $\bar{M}'$, which means by definition, transverse to every simplex of $\bar{M}'$. Write $\bar{M}'_i \subset \bar{M}'$ the jiggled image of $\bar{M}_i$ $(i = 0, 1)$. Thanks to the transversality of F and $\bar{M}_1$, one arranges that $\bar{M}_1$ is globally invariant by the jiggling, and thus $\bar{M}'_1 = \bar{M}_1$ remains a smooth submanifold in E .

Thurston's method consists in applying a homotopy to F , in a small neighborhood of $\bar{M}'$, relative to $\bar{M}'_0$, among the plane fields transverse to $\bar{M}'$, to make F integrable in a neighborhood of $\bar{M}'$. Then, F will induce on $\bar{M}_1$ the seeked foliation.

The homotopy is realized simplex after simplex, climbing up a collapsing of $\bar{M}'$ onto $\bar{M}'_0$ ("inflation") . The heart of the construction is the inflation process: given a $(q+k)$-simplex α of $\bar{M}'$ and a "free" hyperface $\beta \subset \alpha$ s.t. F is already integrable in a neighborhood of $\lambda := (\partial\alpha) \setminus \beta$, one homotopes F relatively to λ , to an integrable field in the whole of α . In a first time, this leaves in the interior of α an unfoliated subset ("hole") diffeomorphic to $\boldsymbol{D}^2 \times \boldsymbol{S}^{k-2} \times \boldsymbol{D}^q$. This hole is filled in a second time. (For $q = 1$, things are a little more complicated: the hole needs to be extended before we fill it).

To prove the *normal* h-principle, write $N := j(\nu\xi)$, a q-plane field on M . Consider a tubular neighborhood T of $\bar{M}_0$ in E . Write $\pi : T \to M$ the projection, $\ker \pi$ the plane field in T tangential to the fibres, and consider $\pi^*(N)$, a foliation on T .

In the beginning, T is a small tubular neighborhood of $\bar{M}_0$ in E . As the inflation process goes, we extend T by successive isotopies of embeddings in E s.t. T remains a small neighborhood of the union of the simplicies where F has been made integrable; and moreover:

- $\ker \pi$ and $\pi^*(N)$ are transverse to $\bar{M}'$;

- $\ker\pi \subset F$ at every point of every free simplex of $\bar{M}'$;
- $\pi^*(N) \pitchfork F$ except in the holes;
- $\pi^*(N) \pitchfork (\boldsymbol{D}^2 \times \boldsymbol{S}^{k-2} \times t)$ in every hole, and for every $t \in \boldsymbol{D}^q$.

It is better not to fill the holes during the inflation. The holes propagate. At the end, F defines on $\bar{M}'$ a Γ_q-structure with holes. Each hole has the form

$$\boldsymbol{D}^2 \times \boldsymbol{S}^{k-2} \times \boldsymbol{D}^{n+1-k} \times \boldsymbol{D}^q$$

where $n := \dim M$, and meets $\bar{M}_1$ on

$$\boldsymbol{D}^2 \times \boldsymbol{S}^{k-2} \times \boldsymbol{S}^{n-k} \times \boldsymbol{D}^q$$

In restriction to $\bar{M}_1$, outside the holes, F defines a foliation normal to $\pi^*(N)$. The projection of $F|\bar{M}_1$ through π is a foliation on M transverse to N , except in the holes.

Then, we can fill the holes by the classical way, obtaining a foliation on M , normal to N but in the holes: this is the normal h-principle.

In case N is an *integrable* q-plane field on M , we can arrange that in each hole, N coincides with the $\boldsymbol{D}^q$-fibres.

Assume moreover that N is tangential to the fibres of a fibration $M \to B$. Then, we can fill each hole by some suspension, at the price of a surgery on B : this leads to some *bordism equivalence* between the classifying space $BDiff_c(\boldsymbol{R}^q)$ for foliated bundles, and the Thom space of $B\bar{\Gamma}_q$; and then, through the Atiyah-Hirzebruch spectral sequence, to the Mather-Thurston homology equivalence.

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