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Rotation number and actions of the modular group on the circle

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1. Introduction

Let Σ be a connected and oriented two dimensional orbifold with empty boundary and negative Euler characteristic $\chi(\Sigma) < 0$. We consider the space $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(\mathrm{S}^1))$ of homomorphisms from $\pi_1(\Sigma)$ to $\mathrm{Homeo}_+(\mathrm{S}^1)$ with the compact-open topology. Let $\phi \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(\mathrm{S}^1))$.

When Σ is a closed surface, we have the Euler number $\mathrm{eu}(\phi) \in \mathbb{Z}$ of ϕ and Milnor-Wood inequality ([7], [10])

$$|\,\mathrm{eu}(\phi)| \leq |\chi(\Sigma)|$$

holds. Matsumoto [6] showed that $|\,\mathrm{eu}(\phi)| = |\chi(\Sigma)|$ if and only if ϕ is semi-conjugate to an injective homomorphism onto a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{Homeo}_+(\mathrm{S}^1)$, which is the holonomy representation of a hyperbolic structure on Σ (we call such a homomorphism a hyperbolization of Σ).

When Minakawa [8] dealt with the case where Σ is compact and has cone points. He defined the Euler number $\mathrm{eu}(\phi) \in \mathbb{Q}$ of ϕ by

$$\mathrm{eu}(\phi) = \frac{\mathrm{eu}(\phi|_\Gamma)}{[\pi_1(\Sigma) \colon \Gamma]},$$

where Γ is a torsion-free subgroup of $\pi_1(\Sigma)$ of finite index, and generalized the above results.

For the case where Σ is a noncompact surface of finite type. Burger, Iozzi and Wienhard [1] introduced the bounded Euler number $\mathrm{eu}^b(\phi) \in \mathbb{R}$ of ϕ by using bounded cohomology and generalized Milnor-Wood inequality and the above result of Matsumoto.

In this talk we deal with the case where Σ is noncompact and has cone points. In particular, we consider Milnor-Wood type inequality on each connected component of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(\mathrm{S}^1))$.

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2. Bounded Euler number

Let Σ be a noncompact, connected and oriented two dimensional orbifold with cone points. For $\phi \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_+(\mathrm{S}^1))$, we define the bounded Euler number $\mathrm{eu}^b(\phi) \in \mathbb{R}$ of ϕ by

$$\mathrm{eu}^b(\phi) = \frac{\mathrm{eu}^b(\phi|_\Gamma)}{[\pi_1(\Sigma) \colon \Gamma]},$$

where Γ is a torsion-free subgroup of $\pi_1(\Sigma)$ of finite index. The bounded Euler number has the following properties.

Proposition 2.1. *(1) We have*

$$\chi(\Sigma) \leq \mathrm{eu}^b(\phi) \leq -\chi(\Sigma). \tag{2.2}$$

Furthermore $\mathrm{eu}^b(\phi) = \pm\chi(\Sigma)$ *if and only if* ϕ *is semi-conjugate to a hyperbolization of* Σ.
(2) Suppose that $\Sigma = \Sigma_{g,n}(q_1 \ldots, q_m)$*, an orbifold whose underlying space is a surface of genus* g *with* p *punctures with* m *cone points of order* $q_1, \ldots, q_m$*. Then under the presentation*

$$\pi_1(\Sigma) = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n, d_1, \ldots, d_m \colon d_k^{q_k}, k = 1, \ldots, m, \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{n} c_j \prod_{k=1}^{m} d_k \rangle,$$

we have

$$\mathrm{eu}^b(\phi) = \widetilde{\mathrm{rot}}(\prod_{i=1}^{g}[\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)}] \prod_{j=1}^{n} \widetilde{\phi(c_j)} \prod_{k=1}^{m} \widetilde{\phi(d_k)}) - \sum_{j=1}^{n} \widetilde{\mathrm{rot}}(\widetilde{\phi(c_j)}) - \sum_{k=1}^{m} \widetilde{\mathrm{rot}}(\widetilde{\phi(d_k)}),$$

where $\tilde{g} \in \widetilde{\mathrm{Homeo}}_+(\mathrm{S}^1)$ *is a lift of* $g \in \mathrm{Homeo}_+(\mathrm{S}^1)$ *and* $\widetilde{\mathrm{rot}} \colon \widetilde{\mathrm{Homeo}}_+(\mathrm{S}^1) \to \mathbb{R}$ *is the translation number.*

Remark 2.3. We make several remarks on the case where $\Sigma = \Sigma_{0,1}(q_1, q_2)$ with $\dfrac{1}{q_1} + \dfrac{1}{q_2} < 1$.
(1) The equality $\mathrm{eu}^b(\phi) = \pm\chi(\Sigma_{0,1}(q_1, q_2))$ can be characterized by rotation numbers without translation numbers. Indeed $\mathrm{eu}^b(\phi) = \pm\chi(\Sigma_{0,1}(q_1, q_2))$ if and only if $(\mathrm{rot}(\phi(c_1)), \mathrm{rot}(\phi(d_1)), \mathrm{rot}(\phi(d_2))) = \left(0, \pm\dfrac{1}{q_1}, \pm\dfrac{1}{q_2}\right)$.

(2) There exists $\phi \in \mathrm{Hom}(\pi_1(\Sigma_{0,1}(q_1, q_2)), \mathrm{Diff}^{\omega}_{+}(\mathrm{S}^1))$ such that $\phi(c)$ is topologically conjugate to a parabolic Möbius transformation and ϕ has an exceptional minimal set. This makes a contrast to the case of closed surface groups [3]. Such a homomorphism is obtained by taking ϕ so that $\phi([a, b])$ has more than two fixed points. If ϕ were minimal, then it is topologically conjugate to a hyperbolization of $\Sigma_{0,1}(2, 3)$ of finite area and hence for every $g \in \pi_1(\Sigma_{0,1}(2, 3))$, $\phi(g)$ has at most two fixed points.

(3) There exists $\phi \in \mathrm{Hom}(\pi_1(\Sigma_{0,1}(2, 3)), \mathrm{Diff}^{\omega}_{+}(\mathrm{S}^1))$ such that ϕ is topologically conjugate to a hyperbolization of $\Sigma_{0,1}(2, 3)$ of finite area but they are not C^1-conjugate. Note that a hyperbolization of $\Sigma_{0,1}(2, 3)$ of finite area is unique up to conjugate in $\mathrm{PSL}(2, \mathbb{R})$. This also makes a contrast to the case of closed surface groups [4]. Existence of such a homomorphism is established by checking that we can deform $\phi \in \mathrm{Hom}(\pi_1(\Sigma_{0,1}(2, 3)), \mathrm{Diff}^{\omega}_{+}(\mathrm{S}^1))$ so that ϕ is kept topologically conjugate to a hyperbolization of $\Sigma_{0,1}(2, 3)$ of finite area and the derivative of $\phi([a, b])$ at the attracting fixed point varies.

3. Extremals on connected components

Let $m, n \geq 1$ and $\Sigma = \Sigma_{g,n}(q_1, \ldots, q_m)$. For integers $p_1, \ldots, p_m$, we put

$$H_{g,n}\left(\frac{p_1}{q_1}, \ldots, \frac{p_m}{q_m}\right) = \left\{\phi \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_{+}(\mathrm{S}^1))\colon\ \mathrm{rot}(\phi(d_k)) = \frac{p_k}{q_k}, k = 1, \ldots, m\right\}.$$

Since $n \geq 1$, the subset $H_{g,n}(\frac{p_1}{q_1}, \ldots, \frac{p_m}{q_m})$ is a connected component of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Homeo}_{+}(\mathrm{S}^1))$. The inequality (2.2) is not optimal on each connected component $H_{g,n}\left(\frac{p_1}{q_1}, \ldots, \frac{p_m}{q_m}\right)$. We can obtain the optimal inequality by Proposition 2.1 (2) and results of Jankins, Neumann [5] and Naimi [9] (see also [2] for more general study). For example, when $\Sigma = \Sigma_{0,1}(2, 3)$, we have

$$\frac{1}{5}\chi(\Sigma) \leq \mathrm{eu}^b(\phi) \leq -\chi(\Sigma)$$

on $H_{0,1}\left(\frac{1}{2}, \frac{1}{3}\right)$ and

$$\chi(\Sigma) \leq \mathrm{eu}^b(\phi) \leq -\frac{1}{5}\chi(\Sigma)$$

on $H_{0,1}\left(\frac{1}{2}, -\frac{1}{3}\right)$. Note that $\phi \in H\left(\frac{1}{2}, \pm\frac{1}{3}\right)$ satisfies $\mathrm{eu}^b(\phi) = \pm\frac{1}{5}\chi(\Sigma)$ if and only if $\mathrm{rot}(c_1) = \pm\frac{1}{5}$. In this case, we have the following result.

Theorem 3.1. *If $\Sigma = \Sigma_{0,1}(2,3)$ and $\phi \in H_{0,1}\left(\frac{1}{2}, \pm\frac{1}{3}\right)$ satisfies $\mathrm{eu}^b(\phi) = \pm\frac{1}{5}\chi(\Sigma)$, then ϕ is semi-conjugate to a 5-fold covering of a hyperbolization of Σ.*

Remark 3.2. Theorem 3.1 cannot be generalized straightforward when we change Σ and $(p_1, \ldots, p_m)$. For example, when $\Sigma = \Sigma_{0,1}(2,7)$, we have

$$\chi(\Sigma) \leq \mathrm{eu}^b(\phi) \leq -\frac{3}{25}\chi(\Sigma)$$

on $H_{0,1}(\frac{1}{2}, \frac{1}{7})$ and $\phi \in H_{0,1}(\frac{1}{2}, \frac{1}{7})$ with $\mathrm{eu}^b(\phi) = -\frac{3}{25}\chi(\Sigma)$ is not semi-conjugate to a finite covering of a hyperbolization of Σ.

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