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On codimension two contact embeddings

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1. Introduction and the statements of the results

We study codimension two contact embeddings in the odd dimensional Euclidean space. Let (M^{2n-1}, ξ) be a closed contact manifold and (N^{2m-1}, η) be a co-oriented contact manifold. An embedding $f : M^{2n-1} \to N^{2m-1}$ is said to be a contact embedding if $f_*(TM^{2n-1}) \cap \eta|_{f(M^{2n-1})} = f_*\xi$. Note that ξ must be co-orientable since $f^*\beta$ is a global defining 1-form of ξ, where β is a global defining 1-form of η. For given (M^{2n-1}, ξ), we would like to know whether there exists a contact embedding of (M^{2n-1}, ξ) in $(\mathbb{R}^{2n+1}, \eta_0)$, where η_0 is the standard contact structure on $\mathbb{R}^{2n+1}$. It is equivalent to the existence of contact embeddings of (M^{2n-1}, ξ) in the $(2n+1)$-sphere with the standard contact structure. We see that the first Chern class is an obstruction for the existence of such an embedding.

Theorem 1.1. *If a closed contact manifold (M^{2n-1}, ξ) is a contact submanifold of a co-oriented contact manifold (N^{2n+1}, η) satisfying the condition $H^2(N^{2n+1}; \mathbb{Z}) = 0$, then the first Chern class $c_1(\xi)$ vanishes.*

In particular, there are infinitely many contact 3-manifolds which cannot be embedded in $(\mathbb{R}^5, \eta_0)$ as contact submanifolds. We note that any 3-manifold can be embedded in $\mathbb{R}^5$ by Wall's theorem[16]. We also note that A.Mori[10] constructed a contact immersion of any closed co-orientable contact 3-manifold in $(\mathbb{R}^5, \eta_0)$ and D.Martinez[9] proved that any closed co-orientable contact $(2n+1)$-manifold can be embedded in $(\mathbb{R}^{4n+3}, \eta_0)$ as a contact submanifold. For the existence of contact embeddings of contact 3-manifolds in $(\mathbb{R}^5, \eta_0)$, there are several known examples. Some of them are links of isolated complex surface singularities in $\mathbb{C}^3$. The canonical contact structure on a link is given by the complex tangency, and it is a contact submanifold of (S^5, η_{std}), where η_{std} is the standard contact structure on S^5. Though it is difficult to determine the structure on a link in general, it is done in the cases of the quasi-homogeneous singularities[13] and the cusp singularities[4],[11],[13]. In these cases, the link is the quotient of a cocompact lattice of a Lie group G and the contact structure is invariant under the action of G. Another example is given by A.Mori[12] and Niederkrüger-Presas[14]. They independently constructed a contact embedding of the overtwisted contact structure on S^3 associated to the negative Hopf band in (S^5, η_{std}). In spite of these examples, we do not know whether every contact 3-manifold with $c_1(\xi) = 0$ can be embedded in $(\mathbb{R}^5, \eta_0)$ as a contact submanifold. By Gromov's h-principle, however, we can show the following result.

Theorem 1.2. *If $c_1(\xi) = 0$, we can embed (M^3, ξ) in $\mathbb{R}^5$ as a contact submanifold for some contact structure on $\mathbb{R}^5$.*

2. Preliminary

2.1. The Chern classes of a co-oriented contact structure

Let $(M^{2n-1}, \xi = \ker \alpha)$ be a co-oriented contact structure. Since the 2-form $d\alpha$ induces a symplectic structure on ξ, $(\xi, d\alpha|_\xi)$ is a symplectic vector bundle over M^{2n-1}. Since the conformal class of the symplectic bundle structure does not depend on the choice of α, we define the Chern classes of ξ as the Chern classes of this symplectic vector bundle.

2.2. The conformal symplectic normal bundle of a contact submanifold

Let $(M, \eta_M) \subset (N, \eta = \ker \beta)$ be a contact submanifold. The vector bundle η splits along M into the Whitney sum of the two subbundles

$$\eta|_M = \eta_M \oplus (\eta_M)^\perp,$$

where η_M is the contact plane bundle on M given by $\eta_M = TM \cap \eta|_M$ and $(\eta_M)^\perp$ is the symplectic orthogonal of η_M in $\eta|_M$ with respect to the form $d\beta$. We can identify $(\eta_M)^\perp$ with the normal bundle νM. Moreover, $d\beta$ induces a conformal symplectic structure on $(\eta_M)^\perp$. We call $(\eta_M)^\perp$ the conformal symplectic normal bundle of M in N.

2.3. The Euler class of the normal bundle of an embedding

Let K^k be a closed orientable k-manifold, L^l an orientable l-manifold and $f\colon K^k \to L^l$ an embedding.

Theorem 2.1. *If $H^{l-k}(L^l; \mathbb{Z}) = 0$, the Euler class of the normal bundle of f vanishes.*

Proof. By Theorem 11.3 of [7], the Euler class of the normal bundle of f is the image of the dual cohomology class of K^k by the homomorphism $f^* : H^{l-k}(L^l; \mathbb{Z}) \to H^{l-k}(K^k; \mathbb{Z})$. Thus, if $H^{l-k}(L^l; \mathbb{Z}) = 0$, it vanishes. □

In particular, when $l = k + 2$, the normal bundle is a 2-dimensional trivial vector bundle.

3. Proof of Theorem 1.1

Proof. Let $f : M^{2n-1} \to N^{2n+1}$ be an embedding such that

$$f_*(TM^{2n-1}) \cap \eta|_{f(M^{2n-1})} = f_*\xi.$$

Since $H^2(N^{2n+1}; \mathbb{Z}) = 0$ and the normal bundle of f is 2-dimensional, it is topologically trivial by Theorem 2.1. Since the conformal symplectic structure on 2-dimensional trivial vector bundle is unique, the normal bundle of $f(M^{2n-1})$ is also trivial as a conformal symplectic vector bundle. That is, the vector bundle η splits along $f(M^{2n-1})$ such that

$$\eta|_{f(M^{2n-1})} = \eta_{f(M^{2n-1})} \oplus (\eta_{f(M^{2n-1})})^{\perp},$$

where $\eta_{f(M^{2n-1})} = f_*\xi$ and $(\eta_{f(M^{2n-1})})^{\perp}$ is a trivial symplectic bundle. By the naturality of the first Chern class and the condition $H^2(N^{2n+1}; \mathbb{Z}) = 0$, it follows that $c_1(\eta|_{f(M^{2n-1})}) = f^*c_1(\eta) = 0$. On the other hand, taking the Whitney sum with a trivial symplectic bundle does not change the first Chern class. Thus, $c_1(\eta|_{f(M^{2n-1})}) = c_1(\xi)$ holds. It follws that $c_1(\xi) = 0$. □

4. Proof of Theorem 1.2

4.1. h-principle

We review Gromov's h-principle and prove Propositon 4.4 as a preliminary for the proof of Theorem 1.2.

Definition 4.1. Let N^{2n+1} be an oriented manifold. An almost contact structure on N^{2n+1} is a pair (β_1, β_2) consisting of a global 1-form β_1 and a global 2-form β_2 satisfying the condition $\beta_1 \wedge \beta_2^n \neq 0$.

Remark 4.2. There is another definition. We can define an almost contact structure on N^{2n+1} as a reduction of the structure group of TN^{2n+1} from $\mathrm{SO}(2n+1)$ to $\mathrm{U}(n)$. Since a pair (β_1, β_2) satisfying $\beta_1 \wedge \beta_2^n \neq 0$ can be seen as the cooriented hyperplane field $\ker \beta_1$ with an almost complex structure compatible with the symplectic structure $\beta_2|_{\ker \beta_1}$, the two definitions are equivalent up to homotopy.

Theorem 4.3 (Gromov[2], Eliashberg-Mishachev[1]). *Suppose N^{2n+1} is an open manifold. If there exists an almost contact structure over N^{2n+1}, then there exists a contact structure on N^{2n+1} in the same homotopy class* *of almost contact structures. Moreover if the almost contact structure is already a contact structure on a neighborhood of a compact submanifold $M^m \subset N^{2n+1}$ with $m < 2n$, then we can get a contact structure on N^{2n+1} which coincides with the original one on a small neighborhood of M^m.*

Let $(M^{2n-1}, \xi = \ker \alpha)$ be a closed cooriented contact manifold and M^{2n-1} be embedded in $\mathbb{R}^{2n+1}$. By Theorem 2.1, there exists an embedding

$$F\colon M^{2n-1} \times D^2 \to \mathbb{R}^{2n+1}.$$

The form $\alpha + r^2 d\theta$ induces a contact form β on $U = F(M^{2n-1} \times D^2)$. By Theorem 4.3, in order to extend given contact structure, it is enough to extend it as an almost contact structure. Almost contact structures on N^{2n+1} correspond to sections of the principal $SO(2n+1)/U(n)$ bundle associated with the tangent bundle TN^{2n+1}. In particular, almost contact structures on $\mathbb{R}^{2n+1}$ correspond to smooth maps

$$\mathbb{R}^{2n+1} \to SO(2n+1)/U(n).$$

Thus we get the following proposition.

Proposition 4.4. *We can embed (M^{2n-1}, ξ) in $\mathbb{R}^{2n+1}$ as a contact submanifold for some contact structure, if and only if there exists an embedding $F\colon M^{2n-1} \times D^2 \to \mathbb{R}^{2n+1}$ such that the map $g : M^{2n-1} \to SO(2n+1)/U(n)$ induced by the underlying almost contact structure of $(M^{2n-1} \times D^2, \alpha + r^2 d\theta)$ is contractible.*

Proof. The underlying almost contact structure of $(U, \beta) \subset \mathbb{R}^{2n+1}$ is identified with the map $\tilde{g} : U \to SO(2n+1)/U(n)$ whose restriction to M^{2n-1} is g. We can take an extension of $\tilde{g}$ over $\mathbb{R}^{2n+1}$ if and only if g is contractible. $\square$

4.2. Proof of Theorem 1.2

Proof. There exists an embedding $f\colon M^3 \to \mathbb{R}^5$ [16], and the normal bundle of f is trivial. Thus we can take an embedding $F\colon M^3 \times D^2 \to \mathbb{R}^5$. By Proposition 4.4, it is enough to prove that if $c_1(\xi) = 0$, then there exists an embedding F such that the map $g\colon M^3 \to SO(5)/U(2)$ induced by F is contractible. Let us take a triangulation of M^3 and $M^{(l)}$ be its l dimensional skeleton, i.e.,

$$M^{(0)} \subset M^{(1)} \subset M^{(2)} \subset M^{(3)} = M^3.$$

The condition $c_1(\xi) = 0$ is equivalent to that ξ is a trivial plane bundle over M^3. Hence a trivialization τ of ξ and the Reeb vector field R of α give a trivialization of TM^3. This trivialization of TM^3 and a trivialization ν of the normal bundle νM^3 form a map

$$h\colon M^3 \to SO(5).$$

In other words, h is a trivialization of $T\mathbb{R}^5 \mid_{M^3}$ consisting of R, τ and ν. Composing with the projection $\pi\colon SO(5) \to SO(5)/U(2)$, it induces the map $g = \pi \circ h\colon M^3 \to SO(5)/U(2)$. Thus h is a lift of g. Now we consider whether h is null-homotopic over $M^{(1)}$. In other words, we consider the difference between the spin structures on $T\mathbb{R}^5 \mid_{M^3}$ induced by h and the constant map I_5. Then the obstruction is the Wu invariant $c(f) \in \Gamma_2(M^3)$, where $\Gamma_2(M^3) = \{C \in H^2(M^3;\mathbb{Z}) \mid 2C = 0\}$. The following explanation of the Wu invariant is due to [15]. The Wu invariant is defined for an immersion of the parallelized 3-manifold with trivial normal bundle. A normal trivialization ν of f and the tangent trivialization define a map $\pi_1(M^3) \to \pi_1(SO(5))$, namely an element $\tilde{c}_f$ in $H^1(M^3;\mathbb{Z}_2)$. If we change ν by an element $z \in [M^3, SO(2)] = H^1(M^3;\mathbb{Z})$, then the class $\tilde{c}_f$ changes by $\rho(z)$, where ρ is the mod 2 reduction map $H^1(M^3;\mathbb{Z}) \to H^1(M^3;\mathbb{Z}_2)$. Hence the coset of $\tilde{c}_f$ in $H^1(M^3;\mathbb{Z}_2)/\rho(H^1(M^3;\mathbb{Z}))$ does not depend on ν. The cokernel of ρ is identified with $\Gamma_2(M^3)$ by the canonical map induce by the Bockstein homomorphism. Under this identification, the coset of $\tilde{c}_f$ corresponds to the Wu invariant $c(f) \in \Gamma_2(M^3)$. Now we fix the trivialization of TM^3 formed by τ and R. By Theorem 3.8 of [15], there exists an embedding $f : M^3 \to \mathbb{R}^5$ such that $c(f) = 0$. Moreover, there exists a normal trivialization ν of f such that $\tilde{c}_f = 0 \in H^1(M^3;\mathbb{Z}_2)$. With the embedding f and the normal trivialization ν, the map h is null-homotopic over $M^{(1)}$. Since $\pi_2(SO(5)) = 0$, it is also null-homotopic over $M^{(2)}$ and so is the map $g = \pi \circ h : M^3 \to SO(5)/U(2)$. Since $\pi_3(SO(5)/U(2)) = 0$, g is contractible. This completes the proof of Theorem 1.2. □

5. Examples of codimension 2 contact submanifolds

5.1. Singularity links

Let X be a complex algebraic surface in $\mathbb{C}^3$ with an isolated singularity at the origin 0. The intersection L^3 of X and a sufficiently small sphere S^5_ε is called the link of $(X, 0)$. The canonical contact structure ξ on L^3 is given by $\xi = TL^3 \cap JTL^3$, where J is the standard complex structure on $\mathbb{C}^3$. It is obviously a contact submanifold of (S^5, η_{std}). In the case of quasi-homogeneous singularity and cusp singularity, Neumann[13] showed that there is a one-one correspondence between geometric structures on L^3 and complex analytic structures on $(X, 0)$.

Example 5.1 (Brieskorn singularity). Let $X = \{x^p + y^q + z^r = 0\}$. The link L^3 is a quotient of the Lie group $G = SU(2), Nil^3$ or $\widetilde{SL}(2;\mathbb{R})$, according as the rational number $p^{-1}+q^{-1}+r^{-1}-1$ is positive, zero or negative [8]. Since the canonical contact structure ξ on L^3 is invariant under the action of G, ξ is determined[13].

Example 5.2 (Cusp singularity). Let $X = \{x^p + y^q + z^r + xyz = 0\}$ with $p^{-1}+q^{-1}+r^{-1} < 1$. This singularity is analytically equivalent to a Hilbert modular cusp associated with a quadratic field over $\mathbb{Q}$ [3],[5],[6]. Thus the link L^3 is a hyperbolic mapping torus and has a geometry of the Lie group $G = Sol^3$. ξ is the positive contact structure associated with the Anosov flow on L^3 [4],[11],[13].

5.2. Other examples

Let $(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3)$ be the polar coordinates on $S^5 \subset \mathbb{C}^3$, where

$$(z_1, z_2, z_3) = (r_1 e^{2\pi i\theta_1}, r_2 e^{2\pi i\theta_2}, r_3 e^{2\pi i\theta_3}) \in \mathbb{C}^3,\ S^5 = \left\{r_1^2 + r_2^2 + r_3^2 = 1\right\}.$$

The standard contact form on S^5 is $\alpha_0 = r_1^2 d\theta_1 + r_2^2 d\theta_2 + r_3^2 d\theta_3$. Let $\phi\colon S^5 \to \mathbb{R}^3$ be the projection, where $\phi(r_1, \theta_1, r_2, \theta_2, r_3, \theta_3) = (r_1^2, r_2^2, r_3^2)$. Then the image $\phi(S^5) = \{x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$ is a regular triangle in $\mathbb{R}^3$. It is called the moment polytope $\triangle$. Note that π is a T^3-fibration over Int$\triangle$ and is a T^2-fibration over $\partial\triangle$ except on the three vertices. Choosing a curve c on $\triangle$ and a section over c appropriately, one can get an embedding of a 3-manifold in S^5.

Example 5.3 (Mori's example). Let (S^3, η_{neg}) be the negative overtwisted contact structure associated with the negative Hopf link. Using the moment polytope, A.Mori constructed a deformation of embedded standard contact 3-sphere to (S^3, η_{neg}) in (S^5, ξ_{std}), via the Reeb foliation on S^3 foliated by immersed Legendrian submanifolds of S^5 [12]. Slightly changing this example, we can also see that tight contact structures on the 3-torus can be embedded in (S^5, η_{std}) as contact submanifolds.

Example 5.4 (Furukawa's example). In a similar way, R.Furukawa constructed the contact embeddings of universally tight contact structures on some T^2 bundles over S^1. His examples cover the link of cusp singularities and Brieskorn Nil singularities.

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