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# Minimal sets for foliations

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## 1. Introduction

In this talk, we will discuss recent results in the program to understand the exceptional minimal sets for foliations of codimension $q \geq 1$. The outline of this program is discussed below.

Let $\mathcal{F}$ be a $C^r$-foliation of a compact connected Riemannian manifold $M$, for $r \geq 0$. The leaves of $\mathcal{F}$ are then smoothly immersed submanifolds in $M$ of codimension $q \geq 1$, and each leaf inherits a natural quasi-isometry class of Riemannian metrics.

A closed subset $\mathfrak{M} \subset M$ is *minimal* for $\mathcal{F}$ if for each $x \in \mathfrak{M}$ the leaf $L_x \subset \mathfrak{M}$, and the closure $\overline{L_x} = \mathfrak{M}$. Moreover, if for each transversal $\mathcal{T}_x \subset M$ to $\mathcal{F}$, the intersection $\mathcal{T}_x \cap \mathfrak{M}$ is a totally disconnected set, hence is homeomorphic to a Cantor set as $\mathfrak{M}$ is minimal, then we say that $\mathfrak{M}$ is an *exceptional minimal set*. Here is the general problem.

Problem 1.1. Classify the exceptional minimal sets for $C^r$-foliations, up to homeomorphism (or possibly orbit equivalence), where $q \geq 1$ and $r \geq 0$.

The approach we take to this very broad problem, is to consider an exceptional minimal set $\mathfrak{M} \subset M$ as a smooth foliated space in the sense of [21], or Candel and Conlon [3, Chapter 11], with additional properties.

Definition 1.2. A *matchbox manifold* is a smooth foliated space $\mathfrak{M}$, whose transverse models $\{\mathfrak{X}_i \mid 1 \leq i \leq \nu\}$ for the foliation charts are totally disconnected compact metric spaces. If every leaf of the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$ is dense, we say that $\mathfrak{M}$ is *minimal*, and then each transversal space $\mathfrak{X}_i$ is a clopen set in some Cantor set model $\mathfrak{X}$.

Definition 1.3. A matchbox manifold $\mathfrak{M}$ is *Lipshitz*, if the holonomy transformations defined by parallel transport along paths in the leaves of $\mathcal{F}_{\mathfrak{M}}$ are Lipshitz homeomorphisms with respect to the given metrics on the models spaces $\{\mathfrak{X}_i \mid 1 \leq i \leq \nu\}$.

With this definition, Problem 1.1 can be stated as:

Problem 1.4. Let $\mathfrak{M}$ be a Lipshitz minimal matchbox manifold. Given $r \geq 0$ and $q \geq 1$, when does there exists a compact Riemannian manifold $M$ with $C^r$-foliation of codimension $q$, and a leafwise smooth embedding $\iota_{\mathfrak{M}} \colon \mathfrak{M} \to M$ so that the image is a minimal set for $\mathcal{F}$?

Observe that if such an embedding $\iota_{\mathfrak{M}} \colon \mathfrak{M} \to M$ exists, then every leaf of $\mathcal{F}_{\mathfrak{M}}$ is realized by a leaf of $\mathcal{F}$ in the same quasi-isometry class of leafwise metrics. Thus, a solution to Problem 1.4 implies a solution to the question posed by Cass in [4]. This problem can also be considered as a generalization of the problem posed by McDuff in [20].

## 2. Existence results

There are a wide variety of constructions of classes of minimal matchbox manifolds, and a vast literature on the study of these special classes. For example, the tiling space $\Omega$ of a tiling of $\mathbb{R}^n$ is defined as the closure of the space of tilings obtained via the translation action of $\mathbb{R}^n$, in a suitable metric topology. The assumption that the tiling is repetitive, aperiodic, and has finite local complexity implies that $\Omega$ is locally homeomorphic to a disk in $\mathbb{R}^n$ times a Cantor set [22], and thus is a matchbox manifold. The *Pisot Conjecture* for tilings essentially asks when a particular class of tilings embeds into a generalized Denjoy $C^1$-foliation.

Weak solenoids were introduced by McCord in [19] and Schori in [23], which generalize the classical case of Vietoris solenoids, which fiber over $B = \mathbb{S}^1$. All weak solenoids are matchbox manifolds with leaves of dimension $n$. Their transverse dynamics are always equicontinuous, and for a base manifold $B$ of dimension $n \geq 2$, there are many subtleties.

The Williams solenoids introduced in [25], which are expanding attractors for Axiom A dynamical systems, are defined as the inverse limit of an expanding map on a branched manifold of dimension $n$. The leaves of the expanding foliation defines a matchbox manifold structure for these.

The Ghys-Kenyon construction in [13, 2] yields the *graph matchbox manifolds*, which have many remarkable properties as a class of examples [17]. Lozano-Rojo and Lukina show in [18] that each *generalized Bernoulli shift* yields a graph matchbox manifold with leaves of dimension 2.

Finally, Chapter 11 of the text by Candel and Conlon [3] contains many constructions of foliated spaces, many of which have totally disconnected transverse models, so are matchbox manifolds.

## 3. Non-embedding results

There are two types of non-embedding results for matchbox manifolds. Note that an embedding of $\mathfrak{M}$ as a minimal set for a foliation of a compact manifold $M$ is a fortiori an embedding of $\mathfrak{M}$ into $M$. Clark and Fokkink prove in [5] the following:

**Theorem 3.1.** *Suppose that $\mathfrak{M}$ is homeomorphic to a weak solenoid with leaves of dimension 1, and the Čech cohomology of $\mathfrak{M}$ is not finite dimensional, then $\mathfrak{M}$ cannot be embedded in a compact manifold $M$ of dimension $n+1$. In particular, such $\mathfrak{M}$ cannot be homeomorphic to a minimal set in a codimension-one foliation.*

For higher codimension, obstructions to embedding a continuum such as $\mathfrak{M}$ into a compact manifold $M$ are more delicate, and do not hold in such generality as above; see the discussion in [6]. The known obstructions to a solution to Problem 1.4 in higher codimensions use properties of the dynamics of $\mathfrak{M}$.

The work [1] by Attie and Hurder introduces the notion of the *leaf entropy* for a leaf of a $C^0$-foliation, whose definition extends naturally to the leaves of a foliated space. The work [16] by Hurder and Lukina use the methods of Lukina in [17] to construct examples of graph matchbox manifolds whose leaves have infinite leaf entropy, which yields:

**Theorem 3.2.** *There exists graph matchbox manifolds $\mathfrak{M}$ which cannot be embedded as a minimal set for any $C^1$-foliation of a compact manifold.*

If $\mathfrak{M}$ is a minimal matchbox manifold which embeds as a minimal set of a $C^1$-foliation of a compact manifold, then there exists a metric on the transverse models $\{\mathfrak{X}_i \mid 1 \leq i \leq \nu\}$ for $\mathcal{F}_{\mathfrak{M}}$ such that the holonomy of $\mathcal{F}_{\mathfrak{M}}$ is Lipshitz. In the work [16] we show:

**Theorem 3.3.** *There exists a minimal matchbox manifold $\mathfrak{M}$ for which there does not exist a metric on the transverse models $\{\mathfrak{X}_i \mid 1 \leq i \leq \nu\}$ such that the holonomy of $\mathcal{F}_{\mathfrak{M}}$ is Lipshitz. Thus, each such example cannot be embedded as a minimal set for any $C^1$-foliation.*

Given a finitely-generated, torsion-free group $\Gamma$, and a minimal action by homeomorphisms $\varphi\colon \Gamma\mathfrak{X} \to \mathfrak{X}$ on a Cantor set $\mathfrak{X}$, then the suspension construction yields a minimal matchbox manifold $\mathfrak{M}$ whose transverse holonomy groupoid is given by the action $\varphi$. The results in [5] show that such matchbox manifolds always admit an embedding into a $C^0$-foliation with codimension 2.

PROBLEM 3.4. Let $\varphi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be a minimal action on a Cantor set $\mathfrak{X}$. Find invariants of the action which are obstructions to embedding a matchbox manifold $\mathfrak{M}$ obtained from a suspension of $\varphi$, into a $C^r$-foliation of a compact manifold, for $r \geq 1$.

## 4. Embedding results

The problem of embedding a 1-dimensional matchbox manifold $\mathfrak{M}$ in a $C^r$-flow has two forms. If the dynamics of the flow restricted to $\mathfrak{M}$ are equicontinuous, or equivalently the flow is almost periodic on $\mathfrak{M}$, then $\mathfrak{M}$ must a Vietoris solenoid [24]. The realization of solenoids as minimal sets for flows has an extensive literature (see [7] for a discussion and references).

The other possibility in the 1-dimensional case, is that the dynamics of $\mathfrak{M}$ are transversally expansive. In this case, $\mathfrak{M}$ has a presentation as an inverse limit of branched 1-manifolds, which can be used to give effective criteria for embedding into punctured surfaces, for example as considered in [14].

The case where $\mathfrak{M}$ is minimal with leaf dimension $n \geq 2$ is much more difficult, and few results are known except when such an embedding is part of the data in the construction, such as for the action of a rank-one lattice in a Lie group, acting on its boundary when it is totally disconnected.

In the work [7], the authors' studied the embedding problem for the base $\mathbb{T}^n$, and developed criteria for when $\mathfrak{M}$ has a smooth embedding.

**Theorem 4.1.** *Let $\mathfrak{M}$ be a weak solenoid with base manifold $\mathbb{T}^n$. Then there exists a $C^0$-foliation $\mathcal{F}$ of codimension-$2n$ on a compact manifold with minimal set $\mathfrak{M}$. If a mild restriction of the model of $\mathfrak{M}$ by compact tori is assumed, then it can be realized by a $C^1$-foliation $\mathcal{F}$ of codimension-$2n$.*

## 5. Classification

The study of the exceptional minimal sets for foliations also includes the problem of classification of minimal matchbox manifolds, up to homeomorphism and orbit equivalence. This is work in progress [8, 9, 10, 11, 15].

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