Geometry and Foliations 2013
Komaba, Tokyo, Japan

Dynamical Lagrangian Foliations: Essential nonsmoothness and Godbillon–Vey classes

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1. Introduction

We present 2 results about foliations arising as stable and unstable foliations for a contact Anosov flow. The first gives Lagrangian foliations on 3-manifolds that can not be smoothed in the following sense: They are preserved by a contact Anosov flow and there is no topologically equivalent contact Anosov flow with C^2 stable and unstable foliations. The second, in early development, gives a representation of Godbillon–Vey classes for the invariant foliations of a contact Anosov flow and has promise for alternate proofs of pertinent results.

2. Nonsmooth foliations

There are contact Anosov flows on 3-manifolds whose Anosov splitting is not C^2 and such that the same holds for any topologically equivalent contact Anosov flow. In this sense, then, the invariant (and necessarily Lagragian) foliations cannot be smoothed out. These Anosov flows turn out to have a remarkable range of unconventional properties.

For a contact Anosov flow on a 3-manifold, the invariant (stable and unstable) foliations are $C^{1+\text{Zygmund}}$, i.e., differentiable with Zygmund-regular derivative. Indeed, this holds for the weak-stable and weak-unstable foliations of volume-preserving Anosov flows on 3-manifolds [10].

Definition 2.1. A continuous function $f\colon U \to L$ on an open set $U \subset L'$ in a normed linear space to a normed linear space is said to be *Zygmund-regular* if there is $Z > 0$ such that $\|f(x+h)+f(x-h)-2f(x)\| \leq Z\|h\|$ for all $x \in U$ and sufficiently small $\|h\|$. It is said to be *"little Zygmund"* (or "zygmund") if $\|f(x+h)+f(x-h)-2f(x)\| = o(\|h\|)$. For maps between manifolds these definitions are applied in smooth local coordinates.

The "nonsmooth" in the section title actually refers to "not $C^{1+\text{zygmund}}$," i.e., no more regular than is always known to be the case. For our purposes the following rigidity result by Hurder and others is central.

Theorem 2.2 ([10, 8]). *If a volume-preserving Anosov flow on a 3-manifold has $C^{1+zygmund}$ Anosov splitting, then it is smoothly conjugate to a geodesic flow (or a suspension).*

To produce examples of contact Anosov flows whose invariant foliations are not $C^{1+\mathrm{zygmund}}$ and such that the same holds for any topologically equivalent contact Anosov flow, it thus suffices to construct contact Anosov flows that are not topologically equivalent to any geodesic flow.

The novelty is that these are *contact* flows, and the novelty of the method (due to Foulon) is to refine previous surgery methods to preserve the existence of a contact structure. The surgery is a Dehn surgery in a knot neighborhood, and in our context the knot should be of the following type.

Definition 2.3. A *Legendrian curve* in a contact manifold is a curve tangent to the contact structure at every point. In the presence of a contact Anosov flow, a Legendrian curve (which is by construction transverse to the flow) is said to be *E-transverse* if it is also transverse to both the strong stable and strong unstable subbundles E^- and E^+ of the flow.

Our main result has a rather long statement because these flows have a host of interesting properties, as do the manifolds we obtain.

Theorem 2.4. *A contact Anosov flow φ on a 3-manifold M with an E-transverse Legendrian knot K admits smooth Dehn surgeries that produce new contact Anosov flows. If φ is the geodesic flow on the unit tangent bundle of a negatively curved surface, then these surgeries include the Handel–Thurston surgery [9], in which case the resulting flow has the following properties:*

1. *It acts on a manifold that is not a unit tangent bundle.*
2. *It is not topologically orbit equivalent to an algebraic flow.*
3. *Its weak stable foliation is not transversely projective [1, Théorème A].*
4. *Its Anosov splitting $TM = E^\varphi \oplus E^+ \oplus E^-$ does not have "little Zygmund" (hence not Lipschitz-continuous) derivative (Theorem 2.2).*
5. *Its topological and volume entropies differ, or, equivalently, the measure of maximal entropy is always singular (otherwise it would be up to finite covers smoothly conjugate to a geodesic flow of constant curvature [7]).*

Moreover, there are contact Anosov flows on hyperbolic manifolds: If $M \smallsetminus K$ is a hyperbolic manifold, then all but finitely many of our Dehn surgeries produce a hyperbolic manifold. The resulting contact Anosov flow (and any contact Anosov flow topologically orbit equivalent to it) has the following additional properties.

6. *It is associated with a new example of a quasigeodesic pseudo-Anosov flow (see Definition 2.5, [6], [12, Section 5]).*
7. *It is not quasigeodesic (Definition 2.5).*
8. *Its orbits are geodesics for suitable Riemannian metrics on M.*
9. *Each closed orbit is isotopic to infinitely many others*[1] *[4, Theorem A], [2, Remark 5.1.16, Theorem 5.3.3], [3].*
10. *Only finitely many pairs of closed orbits bound an* embedded *cylinder*[2] *[3].*

DEFINITION 2.5. A *quasigeodesic* curve is one that is efficient, up to a bounded multiplicative distortion, in measuring distances in relative homotopy classes, and a flow is said to be quasigeodesic if all flow lines are quasigeodesics [5].

3. Godbillon–Vey classes for Legendrian foliations

Consider a contact Anosov flow φ^t on a $2m+1$-dimensional manifold (M, A) with invariant splitting $\mathbb{R}X \oplus E^+ \oplus E^-$. We can take $A(X) \equiv 1$, and $A \restriction_{E^+\oplus E^-} = 0$. Then $i_X dA = 0$ on $E^- \cup E^+$ and $dA \restriction_{\mathbb{R}X\oplus E^-} = 0$. E^+ has dimension m and has an unstable volume a. The *normal n-bundle* of a subbundle F of TM is

$$\mathcal{N}_n(F) := \{\omega \in \bigwedge^n (T^*M) \mid \omega(u_1, \dots, u_n) = 0 \text{ whenever } u_i \in F \text{ for any } i\}.$$

For an unstable volume $a\colon M \to \bigwedge^m(E^+)$ define $\alpha \in \mathcal{N}_m(\mathbb{R}X \oplus E^-)$ by $\alpha \restriction_{E^+} = a$.

Proposition 3.1. *If α is C^1, then there is a 1-form β such that $d\alpha = \beta\wedge\alpha$.*

β is as regular as the foliations. If $\beta = 0$ on E^+, then $i_X d\alpha = \beta(X)\alpha$, i.e., $\beta(X)$ is the infinitesimal relative change of the unstable volume under the flow.

DEFINITION 3.2 (Godbillon–Vey classes). Suppose (M, A) is a contact manifold of dimension $2m + 1$. For an A-preserving Anosov flow $\varphi^t\colon M \to$

[1] For algebraic flows, free homotopy (hence isotopy) classes of closed orbits have cardinality at most 2.

[2] This relation is neither transitive nor reflexive. For comparison, isotopy is the equivalence relation of being the boundary components of an *immersed* cylinder.

M with C^2 Anosov splitting, we define the *Godbillon–Vey classes* by $GV_0 = \int_M A \wedge dA^m$,

$$
\begin{aligned}
GV_1 &= \int_M \beta \wedge dA^m \\
GV_2 &= \int_M \beta \wedge d\beta \wedge dA^{m-1} \\
&\vdots \\
GV_{m+1} &= \int_M \beta \wedge d\beta^m
\end{aligned}
$$

Remark 3.3. We will show that the C^2 assumption is not needed.

Lemma 3.4. *The Godbillon–Vey classes are well-defined, independently of the choices of a and β.*

Theorem 3.5. *GV_0 is the contact (or Liouville) volume. GV_1 is the Liouville entropy ($\beta(X)$ measures the relative rate of change of unstable volume, and the time average (hence by ergodicity, the space average) of this is the sum of the positive Lyapunov exponents, which by the Pesin Entropy Formula is the Liouville entropy), and for geodesic flows of surfaces, GV_2 is the usual Godbillon–Vey class (we derive the Mitsumatsu formula).*

We can apply these classes to geometric rigidity of geodesic flows on surfaces. Analogously to a result of Mitsumatsu [11] we have:

Proposition 3.6. $\dfrac{GV_0 GV_2}{(GV_1)^2} \geq 1$ *with equality iff M has constant curvature.*

Proof. We have $\dim E^- = \dim E^+ = 1$. Denote the standard vertical vector field by Y and the standard horizontal vector field by h to get

$$[X,Y] = -h, \qquad [Y,h] = -X, \qquad [X,h] = RY,$$

where R is the curvature. We write the unstable and stable vector fields as $\xi^{\pm} = u^{\pm} Y \pm h$, where $\dot{u}^{\pm} + {u^{\pm}}^2 + R = 0$ (Riccati equation). With $u := -u^-$ we have

$$GV_0 = \int_M A \wedge dA, \quad GV_1 = \int_M uA \wedge dA, \quad GV_2 = \int_M u^2 + 3(\mathcal{L}_Y u)^2 A \wedge dA,$$

so the Cauchy–Schwarz inequality

$$\int_M uA \wedge dA \leq \Big(\int_M u^2 A \wedge dA \Big)^{1/2} \Big(\int_M A \wedge dA \Big)^{1/2}$$

gives

$$GV_1 \leq (GV_2)^{1/2}(GV_0)^{1/2}$$

with equality only if $u \equiv \text{const}$ (and, redundantly, $\mathcal{L}_Y u \equiv 0$), which in turn happens iff M has constant curvature. $\square$

This easily recovers a rigidity result of Hurder and Katok.

Theorem 3.7. *Suppose φ^t and ψ^t are geodesic flows for Riemannian surfaces M and S, respectively, and S has constant curvature -1. If F is a conjugacy that sends the contact form A for φ^t to that for ψ^t, and if the Godbillon–Vey classes match up, i.e., $GV_i = GV_i'$ for $i = 0, 1, 2$, then M and S are isometric.*

Proof. For the constantly curved manifold we have $GV_0' = GV_1' = GV_2' = \text{vol}(S)$, so $\dfrac{GV_0 GV_2}{(GV_1)^2} = 1$, and Theorem 3.6 implies that M has constant curvature. $\square$

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