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Contact structures, deformations and taut foliations

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1. Introduction

In her PhD thesis H. Eynard-Bontemps proved the following theorem:

Theorem 1.1 (Eynard-Bontemps [5]). *Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be smooth oriented taut foliations on a 3-manifold M whose tangent distributions are homotopic as (oriented) plane fields. Then $T\mathcal{F}_0$ and $T\mathcal{F}_1$ are smoothly homotopic through integrable plane fields.*

This raises the question of whether any two taut contact structures that are homotopic as plane fields are also homotopic as taut foliations. An interesting special case of this concerns the path connectedness of the space of horizontal foliations on S^1-bundles (i.e. those that are transverse to the fibers). We provide various examples which show that the answer to both of these questions is negative. One of the main tools are contact perturbations of foliations given by Eliashberg and Thurston [3].

This naturally leads to the problem, first raised by Eliashberg and Thurston (see also [4]), of which (universally tight) contact structures are perturbations of taut or Reebless foliations, which can be answered completely for Seifert fibered spaces over surfaces of genus at least one.

2. Main results

Let $\mathrm{Rep}_e(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1))$ denote the space of holonomy representations of smooth horizontal foliations on an oriented S^1-bundle of Euler class e over a closed, oriented surface Σ_g of genus g.

Theorem 2.1. *The space $Rep_e(\pi_1(\Sigma_g), Diff_+(S^1))$ with fixed Euler class $e \neq 0$ is in general not path connected.*

To prove this theorem one distinguishes path components of the space $\mathrm{Rep}_e(\pi_1(\Sigma_g), \mathrm{Diff}_+(S^1))$ using the isotopy class of contact perturbations approximating the associated suspension foliations. However, care must be taken as the isotopy class of a contact structure approximating a contact structure is in general not well-defined. On the other hand Vogel [8] has shown that the isotopy class of the approximating contact structure is well-defined for foliations without torus leaves, apart from a small list of special cases, although for our applications a relatively simple argument using linear deformations of foliations suffices.

Theorem 2.1 can also be shown using the following extension of a result of Ghys [7], which answers a question posed to us by Y. Mitsumatsu.

Theorem 2.2. *Any representation $\rho \in Rep(\pi_1(\Sigma_g), Diff_+(S^1))$ that lies in the C^0-connected component of an Anosov representation ρ_{An} is itself Anosov. In particular, it is conjugate to a discrete subgroup of a finite covering of $PSL(2,\mathbb{R})$ and is injective.*

Similar ideas yield the following

Theorem 2.3. *There exist infinitely many examples of manifolds admitting taut foliations $\mathcal{F}_0, \mathcal{F}_1$ that are homotopic as foliations but not as taut foliations. Furthermore, the same result holds true for diffeomorphism classes of unoriented foliations.*

Concerning which contact structures can be realised as perturbations of Reeebless/taut foliations, we obtain a characterisation for a large class of Seifert fibered spaces. In order to state this result recall the notion of the *enroulement* (cf. [6]) or *twisting number* $t(\xi)$ of a contact structure ξ on a Seifert fibered space which is defined as the maximal Thurston-Bennequin number of a Legendrian knot that is isotopic to a regular fiber, where this is measured relative to the canonical framing coming from the base. Moreover, a *deformation* of a foliation $\mathcal{F}$ is a smooth family of 2-plane fields $\{\xi_t\}_{t\in[0,1]}$ so that $\xi_0 = T\mathcal{F}$ and ξ_t is a contact structure for $t > 0$.

Theorem 2.4. *Let ξ be a universally tight contact structure on a Seifert fibered space with infinite fundamental group and $t(\xi) \geq 0$, then ξ is isotopic to a deformation of a Reebless foliation. If $g > 0$ and $t(\xi) < 0$, then ξ is isotopic to a deformation of a taut foliation.*

3. Questions

QUESTION 3.1. (1) Is the space $\mathrm{Rep}_e(\pi_1(\Sigma_g), \mathrm{Homeo}_+(S^1))$ of topological S^1-actions of fixed Euler class path connected? A related question is whether the image of

$$\mathrm{Rep}_e(\pi_1(\Sigma_g), \mathrm{Homeo}_+(S^1)) \xrightarrow{e_b} H^2_b(\pi_1(\Sigma_g), \mathbb{R})$$

under the bounded Euler class is path connected (in the weak-* topology).

(2) Does any 3-manifold M with infinite fundamental group that admits universally tight contact structures for both orientations necessarily admit a smooth Reebless/taut foliation? (Note that the existence of universally tight contact structures for both orientations is a necessary condition by [2]).

(3) Are there examples of manifolds for which the space of taut foliations in a given homotopy class has infinitely manifold path components up to diffeomorphism and deformation?

References

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